A Galois–dynamics correspondence for unicritical polynomials

Robin Zhang

Columbia University

Géométrie algébrique, Théorie des nombres et Applications 2021

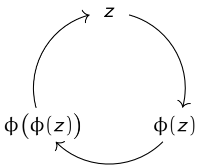
19 August 2021

Overview

1. A dynamical question
   - Rational periodic points
2. Abelian varieties analogue
   - Mordell–Lang conjecture
   - Galois homothety
3. Galois–dynamics correspondence
   - Definition
   - Dynatomic modular curves
   - Rationality perspective
   - Non-existence of quadratic periodic points
   - Irreducibility criterion
   - Case-by-case
   - Further questions

Rational periodic points of quadratic polynomials

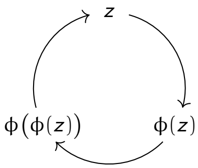
Goal: Characterize the rational periodic points of rational functions $\phi \in \mathbb{Q}(z)$.



A 3-cycle.

Rational periodic points of quadratic polynomials

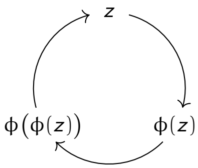
Goal: Characterize the rational periodic points of quadratic polynomials $\phi \in \mathbb{Q}[z]$.



A 3-cycle.

Rational periodic points of quadratic polynomials

Goal: Characterize the rational periodic points of quadratic polynomials $\phi \in \mathbb{Q}[z]$.



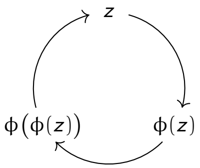
A 3-cycle.

Example (Fixed point)

Let $\phi(z) := z^2 + \frac{1}{4}$.

Rational periodic points of quadratic polynomials

Goal: Characterize the rational periodic points of quadratic polynomials $\phi \in \mathbb{Q}[z]$.



A 3-cycle.

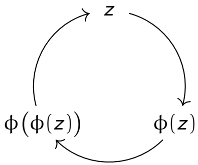
Example (Fixed point)

Let $\phi(z) := z^2 + \frac{1}{4}$.
$z_0 = \frac{1}{2}$ is a fixed point of ϕ.

$$\frac{1}{2} \mapsto \frac{1}{2}$$

Rational periodic points of quadratic polynomials

Goal: Characterize the rational periodic points of quadratic polynomials $\phi \in \mathbb{Q}[z]$.



A 3-cycle.

Example (Fixed point)

Let $\phi(z) := z^2 + \frac{1}{4}$.
$z_0 = \frac{1}{2}$ is a fixed point of ϕ.

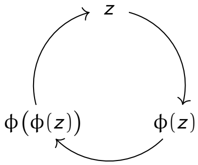
$$\frac{1}{2} \mapsto \frac{1}{2}$$

Example (Period 2)

Let $\phi(z) := z^2 - 1$.

Rational periodic points of quadratic polynomials

Goal: Characterize the rational periodic points of quadratic polynomials $\phi \in \mathbb{Q}[z]$.



A 3-cycle.

Example (Fixed point)

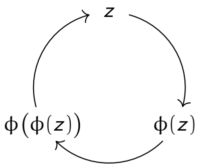
Let $\phi(z) := z^2 + \frac{1}{4}$.
$z_0 = \frac{1}{2}$ is a fixed point of ϕ. $\qquad \frac{1}{2} \mapsto \frac{1}{2}$

Example (Period 2)

Let $\phi(z) := z^2 - 1$.
$\{0, 1\}$ is a 2-cycle of ϕ. $\qquad 0 \mapsto -1 \mapsto 0$
1 is a preperiodic point of ϕ but not a periodic point! $\qquad 1 \mapsto 0 \mapsto -1 \mapsto 0$

Rational periodic points of quadratic polynomials

Goal: Characterize the rational periodic points of quadratic polynomials $\phi \in \mathbb{Q}[z]$.



A 3-cycle.

Example (Fixed point)

Let $\phi(z) := z^2 + \frac{1}{4}$.
$z_0 = \frac{1}{2}$ is a fixed point of ϕ.

$$\tfrac{1}{2} \mapsto \tfrac{1}{2}$$

Example (Period 2)

Let $\phi(z) := z^2 - 1$.
$\{0, 1\}$ is a 2-cycle of ϕ.
1 is a preperiodic point of ϕ but not a periodic point!

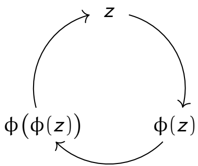
$$0 \mapsto -1 \mapsto 0$$
$$1 \mapsto 0 \mapsto -1 \mapsto 0$$

Example (Period 3)

Let $\phi(z) := z^2 - \frac{29}{16}$.

Rational periodic points of quadratic polynomials

Goal: Characterize the rational periodic points of quadratic polynomials $\phi \in \mathbb{Q}[z]$.



A 3-cycle.

Example (Fixed point)

Let $\phi(z) := z^2 + \frac{1}{4}$.
$z_0 = \frac{1}{2}$ is a fixed point of ϕ.

$$\tfrac{1}{2} \mapsto \tfrac{1}{2}$$

Example (Period 2)

Let $\phi(z) := z^2 - 1$.
$\{0, 1\}$ is a 2-cycle of ϕ.
1 is a preperiodic point of ϕ but not a periodic point!

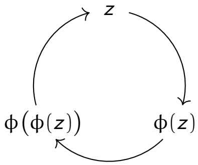
$$0 \mapsto -1 \mapsto 0$$
$$1 \mapsto 0 \mapsto -1 \mapsto 0$$

Example (Period 3)

Let $\phi(z) := z^2 - \frac{29}{16}$.
$\{\frac{5}{4}, -\frac{1}{4}, -\frac{7}{4}\}$ is a 3-cycle of ϕ.

$$\tfrac{5}{4} \mapsto -\tfrac{1}{4} \mapsto -\tfrac{7}{4} \mapsto \tfrac{5}{4}$$

Rational periodic points



A 3-cycle.

Goal: Characterize the rational periodic points of quadratic polynomials $\phi \in \mathbb{Q}[z]$.

$$\phi = a_2 z^2 + a_1 z + a_0 \xrightarrow{\text{linear conjugation}} \phi_c = z^2 + c$$

Linear conjugation preserves dynamical behavior so just study $\phi_c := z^2 + c$ with $c \in \mathbb{Q}$.

Theorem (Fixed point)

Let $\phi_c(z) := z^2 + c \in \mathbb{Q}[z]$.
For any rational λ, ϕ has fixed points λ and $1 - \lambda$ if and only if $c = \lambda - \lambda^2$.

Theorem (Period 2)

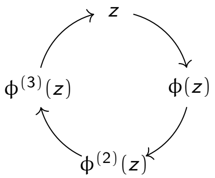
Let $\phi_c(z) := z^2 + c \in \mathbb{Q}[z]$.
For any rational $\lambda \neq -\frac{1}{2}$, ϕ has the 2-cycle $\{\lambda, -1 - \lambda\}$ if and only if $c = -1 - \lambda - \lambda^2$.

Theorem (Period 3, Morton 1992)

Let $\phi_c(z) := z^2 + c \in \mathbb{Q}[z]$.
For any rational $\lambda \notin \{0, 1\}$, ϕ has the 3-cycle $\left\{ \frac{\lambda^3+2\lambda^2+\lambda+1}{2\lambda(\lambda+1)}, \frac{\lambda^3-\lambda-1}{2\lambda(\lambda+1)}, -\frac{\lambda^3+2\lambda^2+3\lambda+1}{2\lambda(\lambda+1)} \right\}$
if and only if $c = -\frac{\lambda^6+2\lambda^5+4\lambda^4+8\lambda^3+9\lambda^2+4\lambda+1}{4\lambda^2(\lambda+1)^2}$.

Quadratic polynomials and $N > 3$

What are the possible periods N?



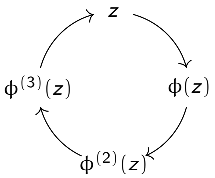
A 4-cycle.

Quadratic polynomials and $N > 3$

What are the possible periods N?

Conjecture (Flynn–Poonen–Schaefer 1997)

There are no quadratic polynomials $\phi \in \mathbb{Q}[z]$ with a rational periodic point of exact period $N \geq 4$.



A 4-cycle.

Quadratic polynomials and $N > 3$

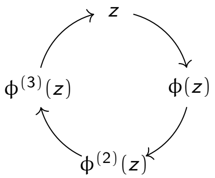
What are the possible periods N?

Conjecture (Flynn–Poonen–Schaefer 1997)

There are no quadratic polynomials $\phi \in \mathbb{Q}[z]$ with a rational periodic point of exact period $N \geq 4$.

Theorem

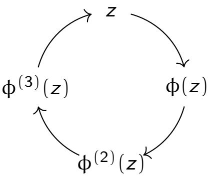
- $N = 4$: Morton (1998)
- $N = 5$: Flynn–Poonen–Schaefer (1997)
- $N = 6$: Stoll (2008) assuming standard conjectures on $J(C_0(6))$



A 4-cycle.

Quadratic polynomials and $N > 3$

What are the possible periods N?



A 4-cycle.

Conjecture (Flynn–Poonen–Schaefer 1997)

There are no quadratic polynomials $\phi \in \mathbb{Q}[z]$ with a rational periodic point of exact period $N \geq 4$.

Theorem

- $N = 4$: Morton (1998)
- $N = 5$: Flynn–Poonen–Schaefer (1997)
- $N = 6$: Stoll (2008) assuming standard conjectures on $J(C_0(6))$

Conjecture

The only quadratic polynomials $\phi \in \mathbb{Q}[z]$ with a quadratic periodic point of exact period $N \geq 5$ is the following example with $\phi_{d,c} = z^2 - \frac{71}{48}$ and $K = \mathbb{Q}(\sqrt{33})$.

- $z_0 = -1 + \frac{\sqrt{33}}{12}$
- $z_1 = -\frac{1}{4} - \frac{\sqrt{33}}{6}$
- $z_2 = -\frac{1}{2} + \frac{\sqrt{33}}{12}$
- $z_3 = -1 - \frac{\sqrt{33}}{12}$
- $z_4 = -\frac{1}{4} + \frac{\sqrt{33}}{6}$
- $z_5 = -\frac{1}{2} - \frac{\sqrt{33}}{12}$

Broader context

Theorem (Northcott 1950)

Given $\phi \in K[z]$, there is a finite number of K-rational preperiodic points of ϕ.

Broader context

Theorem (Northcott 1950)

Given $\phi \in K[z]$, there is a finite number of K-rational preperiodic points of ϕ.

Conjecture (Uniform Boundedness Conjecture, Morton–Silverman 1994)

There is a uniform bound $c(d, [K : \mathbb{Q}])$ on the number of K-rational preperiodic points of $\phi \in K[z]$ of degree d.

Broader context

Theorem (Northcott 1950)

Given $\phi \in K[z]$, there is a finite number of K-rational preperiodic points of ϕ.

Conjecture (Uniform Boundedness Conjecture, Morton–Silverman 1994)

There is a uniform bound $c(d, [K : \mathbb{Q}])$ on the number of K-rational preperiodic points of $\phi \in K[z]$ of degree d.

Much of this can be further generalized to rational functions, morphisms $\phi : \mathbb{P}^n \to \mathbb{P}^n$, morphisms $\phi : A \to A$, ...

Broader context

Theorem (Northcott 1950)

Given $\phi \in K[z]$, there is a finite number of K-rational preperiodic points of ϕ.

Conjecture (Uniform Boundedness Conjecture, Morton–Silverman 1994)

There is a uniform bound $c(d, [K : \mathbb{Q}])$ on the number of K-rational preperiodic points of $\phi \in K[z]$ of degree d.

Much of this can be further generalized to rational functions, morphisms $\phi : \mathbb{P}^n \longrightarrow \mathbb{P}^n$, morphisms $\phi : A \longrightarrow A$, ...

Theorem (Looper 2021)

Assuming a generalized *abc* conjecture,
there is a uniform bound $c(d, [K : \mathbb{Q}])$ on the number of K-rational preperiodic points of $\phi \in K[z]$ of degree d.

A conjecture of Lang

- Let A be a (semi-)abelian variety
- Let Γ be a finitely generated subgroup of $A(\mathbb{C})$
- Let $\overline{\Gamma} := \{x \in A(\mathbb{C}) \mid nx \in \Gamma \text{ for some } n \in \mathbb{N}\}$

A conjecture of Lang

- Let A be a (semi-)abelian variety
- Let Γ be a finitely generated subgroup of $A(\mathbb{C})$
- Let $\overline{\Gamma} := \{x \in A(\mathbb{C}) \mid nx \in \Gamma \text{ for some } n \in \mathbb{N}\}$

Theorem (Mordell–Lang conjecture; Laurent, Raynaud, Hindry, Faltings, Vojta, McQuillan)

Let X be a closed subvariety of A.
Then $X \cap \overline{\Gamma}$ is contained in a finite union of translates of semi-abelian subvarieties of X.

A conjecture of Lang

- Let A be a (semi-)abelian variety
- Let Γ be a finitely generated subgroup of $A(\mathbb{C})$
- Let $\overline{\Gamma} := \{x \in A(\mathbb{C}) \mid nx \in \Gamma \text{ for some } n \in \mathbb{N}\}$

Theorem (Mordell–Lang conjecture; Laurent, Raynaud, Hindry, Faltings, Vojta, McQuillan)

Let X be a closed subvariety of A.
Then $X \cap \overline{\Gamma}$ is contained in a finite union of translates of semi-abelian subvarieties of X.

Theorem (Also Mordell–Lang conjecture)

Let X be a closed integral subvariety of A such that $X \cap \overline{\Gamma}$ is Zariski dense in X.
Then X is the translate of a semi-abelian variety by a point of $\overline{\Gamma}$.

A conjecture of Lang

- Let A be a (semi-)abelian variety
- Let Γ be a finitely generated subgroup of $A(\mathbb{C})$
- Let $\overline{\Gamma} := \{x \in A(\mathbb{C}) \mid nx \in \Gamma \text{ for some } n \in \mathbb{N}\}$

Theorem (Mordell–Lang conjecture; Laurent, Raynaud, Hindry, Faltings, Vojta, McQuillan)

Let X be a closed subvariety of A.
Then $X \cap \overline{\Gamma}$ is contained in a finite union of translates of semi-abelian subvarieties of X.

Theorem (Also Mordell–Lang conjecture)

Let X be a closed integral subvariety of A such that $X \cap \overline{\Gamma}$ is Zariski dense in X.
Then X is the translate of a semi-abelian variety by a point of $\overline{\Gamma}$.

Theorem (Also also Mordell–Lang conjecture (weaker))

Let X be a closed subvariety of A which does not contain any translated (semi-)abelian varieties.
Then $X \cap \overline{\Gamma}$ is finite.

Uniform Mordell–Lang conjecture

For A an abelian variety, there is a uniform version of the Mordell-Lang conjecture.
(Mazur 1986, Rémond 2000, David–Philippon 2007, Dimitrov–Gao–Habegger 2020, Kühne 2021, Gao–Ge–Kühne 2021)

Theorem (Gao–Ge–Kühne 2021)

There exist finitely many translates of abelian subvarieties $x_1 + Y_1, \dots x_n + Y_n \subset X$ such that

$$X \cap \Gamma = \left(\bigcup_{i=1}^{n} (x_i + Y_i) \cap \Gamma \right) \coprod S$$

where S is a finite set and

$$n + \#S \leq c(\dim A, \deg_L X)^{\mathrm{rk}\Gamma + 1}$$

A fact about abelian varieties

Theorem ("Galois homothety", Serre 1985–1986)

Let A be an abelian variety A over a number field k. There exists an $i \geq 1$ such that for all positive integers m, there is a $\sigma_m \in \mathrm{Gal}(\overline{k}/k)$ such that

$$\sigma_m(x) = m^i x$$

for all $\overline{k}$-rational points x on A of finite order coprime to m.

Elements of the Galois group $\mathrm{Gal}(\overline{k}/k)$ "mimic" the action of an iterate of $A \xrightarrow{\text{multiplication by } m} A$.

Galois–dynamics correspondence (GDC)

Definition

Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map over $\mathbb{Q}$, $K/\mathbb{Q}$ be a (nontrivial) finite Galois extension and let N be an integer greater than 1.
The tuple $(\phi, N, K/\mathbb{Q})$ satisfies the *Galois–dynamics correspondence* if and only if for every periodic point $z \in K - \mathbb{Q}$ of ϕ with exact period N, there exists a positive integer $i < N$ and a nontrivial $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ such that

$$\sigma(z) = \phi^{(i)}(z).$$

Elements of the Galois group $\mathrm{Gal}(\overline{k}/k)$ "mimic" the action of an iterate of $\mathbb{P}^1 \xrightarrow{\phi} \mathbb{P}^1$.

Dynatomic modular curves

Let $\phi_{d,c} := z^d + c$ be any unicritical polynomial in $\mathbb{Q}[z]$.
Its periodic points of exact period N are determined by the *dynatomic polynomial* $\Phi_N(z, c)$.

Dynatomic modular curves

Let $\phi_{d,c} := z^d + c$ be any unicritical polynomial in $\mathbb{Q}[z]$.
Its periodic points of exact period N are determined by the *dynatomic polynomial* $\Phi_N(z,c)$.

Periodic points of $\phi_{d,c}$ of period N are solutions of

$$0 = \phi_{d,c}^{(N)}(z) - z$$

Dynatomic modular curves

Let $\phi_{d,c} := z^d + c$ be any unicritical polynomial in $\mathbb{Q}[z]$.
Its periodic points of exact period N are determined by the *dynatomic polynomial* $\Phi_N(z, c)$.

Periodic points of $\phi_{d,c}$ of period N are solutions of

$$0 = \phi_{d,c}^{(N)}(z) - z = \prod_{\ell | N} \Phi_\ell(z, c),$$

Dynatomic modular curves

Let $\phi_{d,c} := z^d + c$ be any unicritical polynomial in $\mathbb{Q}[z]$.
Its periodic points of exact period N are determined by the *dynatomic polynomial* $\Phi_N(z, c)$.

Periodic points of $\phi_{d,c}$ of period N are solutions of

$$0 = \phi_{d,c}^{(N)}(z) - z = \prod_{\ell | N} \Phi_\ell(z, c),$$

$$\Phi_\ell(z, c) := \prod_{m|\ell} (\phi_{d,c}^{(m)}(z) - z)^{\mu(\ell/m)} \in \mathbb{Q}[z, c] \qquad \text{(Möbius inversion)}$$

Dynatomic modular curves

Let $\phi_{d,c} := z^d + c$ be any unicritical polynomial in $\mathbb{Q}[z]$.
Its periodic points of exact period N are determined by the *dynatomic polynomial* $\Phi_N(z, c)$.

Periodic points of $\phi_{d,c}$ of period N are solutions of

$$0 = \phi_{d,c}^{(N)}(z) - z = \prod_{\ell | N} \Phi_\ell(z, c),$$

$$\Phi_\ell(z, c) := \prod_{m | \ell} (\phi_{d,c}^{(m)}(z) - z)^{\mu(\ell/m)} \in \mathbb{Q}[z, c] \qquad \text{(Möbius inversion)}$$

Dynatomic modular curves

Let $\phi_{d,c} := z^d + c$ be any unicritical polynomial in $\mathbb{Q}[z]$.
Its periodic points of exact period N are determined by the *dynatomic polynomial* $\Phi_N(z,c)$.

Periodic points of $\phi_{d,c}$ of period N are solutions of

$$0 = \phi_{d,c}^{(N)}(z) - z = \prod_{\ell | N} \Phi_\ell(z,c),$$

$$\Phi_\ell(z,c) := \prod_{m|\ell} (\phi_{d,c}^{(m)}(z) - z)^{\mu(\ell/m)} \in \mathbb{Q}[z,c] \qquad \text{(Möbius inversion)}$$

$C_1(N) :=$ smooth projective model of the affine curve defined by $\{\Phi_N(z,c) = 0\}$
$C_0(N) :=$ quotient curve $C_1(N)/\langle (z,c) \mapsto (\phi_{d,c}(z), c)\rangle$
Tower of covers: $C_1(N) \to C_0(N) \to \mathbb{P}_1$

Dynatomic modular curves

Let $\phi_{d,c} := z^d + c$ be any unicritical polynomial in $\mathbb{Q}[z]$.
Its periodic points of exact period N are determined by the *dynatomic polynomial* $\Phi_N(z, c)$.

Periodic points of $\phi_{d,c}$ of period N are solutions of

$$0 = \phi_{d,c}^{(N)}(z) - z = \prod_{\ell \mid N} \Phi_\ell(z, c),$$

$$\Phi_\ell(z, c) := \prod_{m \mid \ell} (\phi_{d,c}^{(m)}(z) - z)^{\mu(\ell/m)} \in \mathbb{Q}[z, c] \qquad \text{(Möbius inversion)}$$

$C_1(N) :=$ smooth projective model of the affine curve defined by $\{\Phi_N(z, c) = 0\}$
$C_0(N) :=$ quotient curve $C_1(N)/\langle (z, c) \mapsto (\phi_{d,c}(z), c) \rangle$
Tower of covers: $C_1(N) \to C_0(N) \to \mathbb{P}_1$

$$(\phi_{d,c}, \text{periodic point of exact period } N) \longleftrightarrow K\text{-rational point of } C_1(N)$$
$$(\phi_{d,c}, \text{cycle of exact period } N) \longleftrightarrow K\text{-rational point of } C_0(N)$$
$$c \longleftrightarrow K\text{-rational point of } \mathbb{P}_1$$

Dynatomic modular curves

Let $\phi_{d,c} := z^d + c$ be any unicritical polynomial in $\mathbb{Q}[z]$.
Its periodic points of exact period N are determined by the *dynatomic polynomial* $\Phi_N(z, c)$.

Periodic points of $\phi_{d,c}$ of period N are solutions of

$$0 = \phi_{d,c}^{(N)}(z) - z = \prod_{\ell | N} \Phi_\ell(z, c),$$

$$\Phi_\ell(z, c) := \prod_{m|\ell} (\phi_{d,c}^{(m)}(z) - z)^{\mu(\ell/m)} \in \mathbb{Q}[z, c] \qquad \text{(Möbius inversion)}$$

$C_1(N) :=$ smooth projective model of the affine curve defined by $\{\Phi_N(z, c) = 0\}$
$C_0(N) :=$ quotient curve $C_1(N)/\langle (z, c) \mapsto (\phi_{d,c}(z), c) \rangle$
Tower of covers: $C_1(N) \to C_0(N) \to \mathbb{P}_1$

$$(\phi_{d,c}, \text{periodic point of exact period } N) \longleftrightarrow K\text{-rational point of } C_1(N)$$
$$(\phi_{d,c}, \text{cycle of exact period } N) \longleftrightarrow K\text{-rational point of } C_0(N)$$
$$c \longleftrightarrow K\text{-rational point of } \mathbb{P}_1$$

Theorem (Douady–Hubbard 1985, Bousch 1992, Lau–Schleicher 1994, Morton 1996)

$C_1(N)$ is smooth and geometrically irreducible (with $\Phi_N(z, t)$ irreducible in $\mathbb{C}(t)[z]$).

Rationality of quadratic periodic points

Proposition (Z. 2021)

Suppose $K/\mathbb{Q}$ is a quadratic number field and $(\phi_{d,c}, N, K/\mathbb{Q})$ satisfies the Galois–dynamics correspondence. Then:

- For any N-cycle $\{z_0, \ldots, z_{N-1}\}$ of $\phi_{d,c}$ in K, its trace $\sum_{i=1}^{N-1} z_i$ is rational.
- Furthermore, if N is odd then every N-periodic point of $\phi_{d,c}$ is rational.

Rationality of quadratic periodic points

Proposition (Z. 2021)

Suppose $K/\mathbb{Q}$ is a quadratic number field and $(\phi_{d,c}, N, K/\mathbb{Q})$ satisfies the Galois–dynamics correspondence. Then:

- For any N-cycle $\{z_0, \ldots, z_{N-1}\}$ of $\phi_{d,c}$ in K, its trace $\sum_{i=1}^{N-1} z_i$ is rational.
- Furthermore, if N is odd then every N-periodic point of $\phi_{d,c}$ is rational.

Corollary

Let $K/\mathbb{Q}$ be a quadratic number field. If $(\phi_{d,c}, N, K/\mathbb{Q})$ satisfies the Galois–dynamics correspondence, then the fiber above $c \in \mathbb{P}^1(\mathbb{Q})$ on $C_1(N)(K)$ maps to a point in $C_0(N)(\mathbb{Q})$.

$$K\text{-rational points of } C_1(N) \overset{\text{GDC}}{\dashrightarrow} \mathbb{Q}\text{-rational points of } C_0(N)$$

GDC for $d = 2$ and small N

Theorem (Z. 2021)

For the quadratic polynomial $\phi_{2,c}(z) = z^2 + c$ with rational coefficients and all nontrivial Galois extensions $K/\mathbb{Q}$, $(\phi_{2,c}, N, K/\mathbb{Q})$ satisfies the Galois–dynamics correspondence in the following cases:

- $N = 2$: all $c \in \mathbb{Q}$;
- $N = 3$: all $c \in \mathbb{Q}$;
- $N = 4$:
    - $[K : \mathbb{Q}] = 2$: all $c \in \mathbb{Q}$;
    - $[K : \mathbb{Q}] > 2$: all $c \notin \left\{ \frac{-s^3-2s+4}{4s} \middle| s \in \mathbb{Q}^\times \right\}$;
- $N = 5, 6, 7$, or 9: all $c \in \mathbb{Q}$ outside of specified finite sets $\Sigma_{2,N}$.

GDC for $d = 2$ and small N

Theorem (Z. 2021)

For the quadratic polynomial $\phi_{2,c}(z) = z^2 + c$ with rational coefficients and all nontrivial Galois extensions $K/\mathbb{Q}$, $(\phi_{2,c}, N, K/\mathbb{Q})$ satisfies the Galois–dynamics correspondence in the following cases:

- $N = 2$: all $c \in \mathbb{Q}$;
- $N = 3$: all $c \in \mathbb{Q}$;
- $N = 4$:
  - $[K : \mathbb{Q}] = 2$: all $c \in \mathbb{Q}$;
  - $[K : \mathbb{Q}] > 2$: all $c \notin \left\{ \frac{-s^3-2s+4}{4s} \middle| s \in \mathbb{Q}^\times \right\}$;
- $N = 5, 6, 7$, or 9: all $c \in \mathbb{Q}$ outside of specified finite sets $\Sigma_{2,N}$.

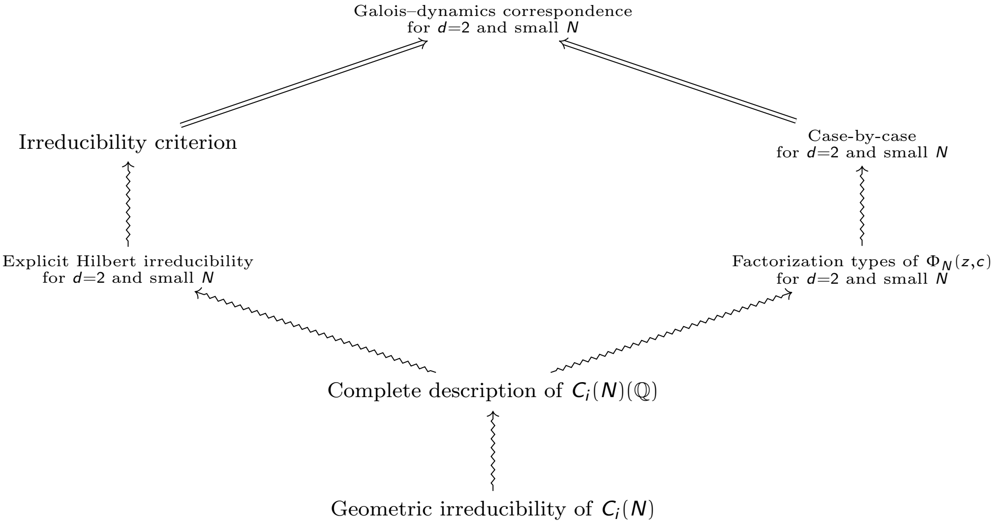
Corollary (N=5)

Let $K/\mathbb{Q}$ be a quadratic number field. There is a specified finite set $\Sigma_{2,5}$ such that if $c \in \mathbb{Q} - \Sigma_{2,5}$, then $\phi_{2,c}$ has no periodic points of exact period 5 in K.

Corollary (N=6)

Let $K/\mathbb{Q}$ be a quadratic number field. Assuming standard conjectures on $L(J(C_0(6)), s)$, there is a specified finite set $\Sigma_{2,6}$ such that if $c \in \mathbb{Q} - \Sigma_{2,6}$, then $\phi_{2,c}$ has no periodic points of exact period 6 in K.

Proof overview



Irreducibility criterion

Proposition (Vivaldi–Hatjispyros 1992)

If $\Phi_N(z, c)$ is irreducible in $\mathbb{Q}[z]$, then $(\phi, N, K/\mathbb{Q})$ satisfies the Galois–dynamics correspondence for all nontrivial Galois extensions $K/\mathbb{Q}$.

Irreducibility criterion

Proposition (Vivaldi–Hatjispyros 1992)

If $\Phi_N(z,c)$ is irreducible in $\mathbb{Q}[z]$, then $(\phi, N, K/\mathbb{Q})$ satisfies the Galois–dynamics correspondence for all nontrivial Galois extensions $K/\mathbb{Q}$.

There are infinite families where $\Phi_N(z,c)$ is reducible in $\mathbb{Q}[z]$:

- $N = 3$ and any c
- $N > 2$ and $c = -2$
- $2^N - 1$ is not a Mersenne prime and $c = 0$

Irreducibility criterion

Proposition (Vivaldi–Hatjispyros 1992)

If $\Phi_N(z, c)$ is irreducible in $\mathbb{Q}[z]$, then $(\phi, N, K/\mathbb{Q})$ satisfies the Galois–dynamics correspondence for all nontrivial Galois extensions $K/\mathbb{Q}$.

There are infinite families where $\Phi_N(z, c)$ is reducible in $\mathbb{Q}[z]$:

- $N = 3$ and any c
- $N > 2$ and $c = -2$
- $2^N - 1$ is not a Mersenne prime and $c = 0$

By the irreducibility of $\Phi_N(z, t)$ in $\mathbb{C}(t)[z]$ and Hilbert's irreducibility theorem:

Proposition

For each integer $d, N > 1$, there is a thin subset $\Sigma_{d,N} \subset \mathbb{P}^1(\mathbb{Q})$ such that for all rational c not in $\Sigma_{d,N}$:

- $\Phi_N(z, c)$ is irreducible in $\mathbb{Q}[z]$,
- The Galois group G_N of $\Phi_N(z, t)$ over $\mathbb{Q}(t)$ is isomorphic to the Galois group $G_{N,c}$ of its specialization $\Phi_N(z, c)$ over $\mathbb{Q}$.

Irreducibility criterion

Proposition (Vivaldi–Hatjispyros 1992)

If $\Phi_N(z,c)$ is irreducible in $\mathbb{Q}[z]$, then $(\phi, N, K/\mathbb{Q})$ satisfies the Galois–dynamics correspondence for all nontrivial Galois extensions $K/\mathbb{Q}$.

There are infinite families where $\Phi_N(z,c)$ is reducible in $\mathbb{Q}[z]$:

- $N = 3$ and any c
- $N > 2$ and $c = -2$
- $2^N - 1$ is not a Mersenne prime and $c = 0$

By the irreducibility of $\Phi_N(z,t)$ in $\mathbb{C}(t)[z]$ and Hilbert's irreducibility theorem:

Proposition

For each integer $d, N > 1$, there is a thin subset $\Sigma_{d,N} \subset \mathbb{P}^1(\mathbb{Q})$ such that for all rational c not in $\Sigma_{d,N}$:

- $\Phi_N(z,c)$ is irreducible in $\mathbb{Q}[z]$,
- The Galois group G_N of $\Phi_N(z,t)$ over $\mathbb{Q}(t)$ is isomorphic to the Galois group $G_{N,c}$ of its specialization $\Phi_N(z,c)$ over $\mathbb{Q}$.

Bousch 1992: $G_N \cong (\mathbb{Z}/N\mathbb{Z}) \wr S_r$ where $rN = \deg \Phi_N$.
Explicit Hilbert irreducibility: Morton 1992 and Krumm 2018 & 2019 give descriptions of $\Sigma_{2,N}$ for small N.

Case-by-case for $d = 2$ and small N

Lemma

Let $K/\mathbb{Q}$ be a (nontrivial) finite Galois extension of degree D and z a periodic point of ϕ_c of exact period $N \geq 2$ in $K - \mathbb{Q}$. If for some $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$,

$$\sigma z = \phi_c^{(i)}(z)$$

then $i = \frac{mN}{g}$ with $0 \leq m < g := \gcd(N, D)$.

This narrows possible indices i when the GDC is satisfied.

Case-by-case for $d = 2$ and small N

Lemma

Let $K/\mathbb{Q}$ be a (nontrivial) finite Galois extension of degree D and z a periodic point of ϕ_c of exact period $N \geq 2$ in $K - \mathbb{Q}$. If for some $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$,

$$\sigma z = \phi_c^{(i)}(z)$$

then $i = \frac{mN}{g}$ with $0 \leq m < g := \gcd(N, D)$.

This narrows possible indices i when the GDC is satisfied.

Use the lemma determining i to deduce that GDC holds for

- $N = 2$: since there is at most one 2-cycle for a given c
- $N = 3$: using the complete determination of $\Phi_3(z, c)$ factorization types (Morton 1992 & Vivaldi–Hatjispyros 1992)
- $N = 4$: due to an explicit parametrization of 4-cycles (Morton 1998, Erkama 2006, Panraksa 2011)

Further questions

- Explicit Hilbert irreducibility for larger N (computationally difficult)
- A less ad-hoc approach to the exceptional cases (i.e. GDC for $c \in \Sigma_{d,N}$)
- GDC for other morphisms and varieties (need irreducibility results for Φ_N)
- Uniform version of GDC closer to the original Galois homothety statement

Further questions

- Explicit Hilbert irreducibility for larger N (computationally difficult)
- A less ad-hoc approach to the exceptional cases (i.e. GDC for $c \in \Sigma_{d,N}$)
- GDC for other morphisms and varieties (need irreducibility results for Φ_N)
- Uniform version of GDC closer to the original Galois homothety statement

- Poisson cru or mahi-mahi?