The twisted forms of a semisimple group over the integral domain of a global function field

Claudia Schoemann

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We study the twisted forms of a semisimple and almost-simple group scheme $\underline{G}$ defined over $\mathrm{Spec}(\mathcal{O}_S)$ with $\mathcal{O}_S$ an integral domain of the global function field $\mathbb{F}_q(C)$ for a projective curve C.

This is joint work with R. Bitan and R. Köhl.

$C :=$ smooth and geometrically connected projective curve over $\mathbb{F}_q$
$K := \mathbb{F}_q(C)$ global field of rational functions on C
$\Omega :=$ set of all closed points on C.

For any $p \in \Omega$ let v_p be the discrete valuation at a prime p of $K, \hat{\mathcal{O}}_p$ the complete valuation ring with respect to v_p, $\hat{K}_p :=$ field of fractions of $\hat{\mathcal{O}}_p, k_p := \hat{\mathcal{O}}_p / p\hat{\mathcal{O}}_p$ the residue field.

Any finite and non-empty set $S \subset \Omega$ gives rise to an integral domain of K, called a Hasse domain

$$\mathcal{O}_S := \{x \in K : v_p(x) \geq 0 \, \forall p \notin S\}.$$

This is a regular and one-dimensional Dedekind domain.

Example 1 : C is the projective line over $\mathbb{F}_q \Rightarrow K = \mathbb{F}_q(x)$.
$S = \{1/x\} \Rightarrow \mathcal{O}_S = \mathbb{F}_q[x]$, $S = \{x, 1/x\} \Rightarrow \mathcal{O}_S = \mathbb{F}_q[x, x^{-1}]$.

Example 2 : Let $C = \{Y^2Z = X^3 + XZ^2\}$ defined over $\mathbb{F}_5$.
Removing $S = \{(0:1:0)\} =$ '∞' $\rightsquigarrow$ affine elliptic curve
$C^{\text{aff}} = \{Y^2 = X^3 + X\}$ with $\mathcal{O}_S = \mathbb{F}_5[x, y]/(y^2 - x^3 - x)$.

Let $\underline{G} :=$ smooth, semisimple and almost-simple $\mathcal{O}_S-$ group,
$\underline{F} := \ker[\underline{G}^{sc} \overset{\pi}{\to} \underline{G}]$ the fundamental group with $(|\underline{F}|, q) = 1$.

A twisted form of $\underline{G}$ is an $\mathcal{O}_S-$ group that is isomorphic to $\underline{G}$ over some finite étale cover of $\mathcal{O}_S$.

Goal : describe explicitely, in terms of invariants of $\underline{F}(\underline{G})$ and the absolute type of the Dynkin diagram of $\underline{G}$, the finite set of all twisted forms of $\underline{G}$, modulo $\mathcal{O}_S-$ isomorphisms.

Approach : describe
1. the $\underline{G}^{\mathrm{ad}}-$ torsors giving the inner forms of $\underline{G}$,
2. the action of the outer automorphisms of $\underline{G}$ on its Dynkin diagram.
$\rightsquigarrow$ twisted forms

Definition

A $\underline{G}$-torsor (in the fppf topology) is a faithfully-flat of finite presentation $\mathcal{O}_S$-scheme P, equipped with a (right) $\underline{G}$-action, such that

$$\mu : P \times_{\mathcal{O}_S} \underline{G} \to P \times_{\mathcal{O}_S} P : (p,g) \mapsto (p,pg)$$

is an isomorphism.

$H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{G}) := \{\underline{G}-$ torsors $\}/\mathcal{O}_S-$ iso. It is finite (BP, 1989).

Similarily we define $H^1(K,G)$ and $H^1_{\mathrm{fl}}(\hat{\mathcal{O}}_p, \underline{G}_p)$ $\forall p \notin S$.

$\rightsquigarrow$ pointed-sets : the $\bullet$ in $H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{G})$ ($H^1(K,G), H^1_{\mathrm{fl}}(\hat{\mathcal{O}}_p, \underline{G}_p)$) is the class of the trivial $\underline{G}$-torsor (trivial G-torsor, trivial $\underline{G}_p$-torsor).

Let P be a right $\underline{G}-$ torsor and $\underline{G}$ acting on itself by conjugation. Then

$$^{P}\underline{G} := P \times_{\mathcal{O}_S} \underline{G} / (ps^{-1}, sgs^{-1}) \ \forall s \in \underline{G}(\mathcal{O}_S)$$

is an affine $\mathcal{O}_S-$ group scheme called the twist of $\underline{G}$ by P. It is locally isomorphic to $\underline{G}$ in the fppf topology : any fiber of it at a prime p of $\mathcal{O}_S$ is isomorphic to $\underline{G}_p := \underline{G} \otimes_{\mathcal{O}_S} \hat{\mathcal{O}}_p$ over some finite flat extension of $\hat{\mathcal{O}}_p$. It's an inner form of $\underline{G}$.

Consider the ring of $S-$integral adèles

$$\mathbb{A}_S := \prod_{p \in S} \hat{K}_p \times \prod_{p \notin S} \hat{\mathcal{O}}_p$$

as a subring of $\mathbb{A}$. (Note that $\mathbb{A} = \bigcup_S \mathbb{A}_S$)

Consider the $S-$class set of $\underline{G}$

$$\mathrm{Cl}_S(\underline{G}) := \underline{G}(\mathbb{A}_S) \backslash \underline{G}(\mathbb{A}) / G(K)$$

where $\underline{G}(\mathbb{A}_S) = \prod_{p \in S} G(\hat{K}_p) \times \prod_{p \notin S} \underline{G}_p(\hat{\mathcal{O}}_p)$.

We have the exact sequence (Y. Nisnevich, 1982)

$$1 \to \mathrm{Cl}_S(\underline{G}) \to H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{G}) \xrightarrow{\lambda} H^1(K, G) \times \prod_{p \notin S} H^1_{\mathrm{fl}}(\hat{\mathcal{O}}_p, \underline{G}_p) \qquad (1)$$

where $\lambda : X \mapsto X \otimes_{\mathcal{O}_S} \mathrm{Spec}(K) \times \prod_{p \notin S} X \otimes_{\mathcal{O}_S} \mathrm{Spec}(\hat{\mathcal{O}}_p)$

By Lang's Lemma we get

$$1 \to \mathrm{Cl}_S(\underline{G}) \to H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{G}) \xrightarrow{\lambda_K} H^1(K, G) \qquad (2)$$

The set of genera of $\underline{G}$ is

$$\text{gen}(\underline{G}) := \{\lambda_K^{-1}([\xi]), [\xi] \in \text{Im}(\lambda_K)\}.$$

Let $[\xi_0] := \lambda_K([\underline{G}])$. Then

$$\lambda_K{}^{-1}([\xi_0])$$

are the classes of $\underline{G}$–torsors that are generically and locally trivial at all points of $\mathcal{O}_S$, and by (2)

$$\text{Cl}_S(\underline{G}) \cong \lambda_K{}^{-1}([\xi_0]),$$

it is the principal genus of $\underline{G}$.

Theorem (G. Harder, 1975 - Local to global principle)

If K is a global function field and G is a simply-connected $K-$group, then $H^1(K, G) = 1$.

Theorem (G. Prasad, 1977 - Strong approximation)

Let K be a global field and S a finite set of places of K. If G is semisimple, simply-connected and almost-simple s.t. $G_S := \prod_{p \in S} G(\hat{K}_p)$ is non-compact, then $G_S G(K)$ is dense in $G(\mathbb{A})$, so $\mathrm{Cl}_S(\underline{G}) = 1$.

Corollary (B.-K.-S.)

Let $\underline{G}$ be a smooth and affine $\mathcal{O}_S$−group with connected fibers. If its generic fiber G is almost-simple, simply-connected and $G_S := \prod_{p \in S} G(\hat{K}_p)$ is non-compact, then $H^1_{ét}(\mathcal{O}_S, \underline{G}) = 1$.

Proof : $\underline{G}$ is simply-connected and K is a function field ($\rightsquigarrow$ no real places), so $H^1(K, G) = 1$.
All $\underline{G}$−torsors are K−isomorphic by (2)

$$1 \to \mathrm{Cl}_S(\underline{G}) \to H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{G}) \xrightarrow{\lambda_K} H^1(K, G)$$

hence $H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{G}) \cong \mathrm{Cl}_S(\underline{G}) = 1$.

□

The universal covering

$$1 \to \underline{F} \to \underline{G}^{\mathrm{sc}} \to \underline{G} \to 1$$

gives rise to the exact sequence

$$H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{G}^{sc}) \to H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{G}) \xrightarrow{\delta} H^2_{\mathrm{fl}}(\mathcal{O}_S, \underline{F})$$

δ is surjective (Douai 1977, Gonzáles-Avilés 2012).

If $\underline{G}$ is almost-simple not of type A, then $H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{G}^{sc}) = 1$ hence δ is also injective.

Lemma

If $\underline{G}$ is almost-simple not of type A then there is an isomorphism of abelian groups

$$H^1_{\text{fl}}(\mathcal{O}_S, \underline{G}) \cong H^2_{\text{fl}}(\mathcal{O}_S, \underline{F}).$$

Let $\underline{F} \cong \text{Res}_{R/\mathcal{O}_S}(\mu_m)$, where $\mu_m := \text{Spec}(\mathcal{O}_S[t])/(t^m - 1)$ and R is a finite étale extension of $\mathcal{O}_S$. $\rightarrow \underline{F}$ is split

We have $H^2_{\text{fl}}(\mathcal{O}_S, \underline{F}) \cong H^2_{\text{fl}}(R, \mu_m)$ by Shapiro's Lemma (SGA 3).

By the Kummer short exact sequence

$$1 \to \mu_m \to \underline{G}_m \overset{x \mapsto x^m}{\to} \underline{G}_m \to 1,$$

identifying

$H^1_{\mathrm{fl}}(R, \underline{G}_m) \cong \mathrm{Pic}(R)$ (cf. Shapiro's Lemma),
$H^2_{\mathrm{fl}}(R, \underline{G}_m) \cong \mathrm{Br}(R)$ (cf. J.S. Milne, Étale cohomology)

and applying flat cohomology we obtain the short and split exact sequence

$$1 \to \mathrm{Pic}(R)/m \to H^2_{\mathrm{fl}}(R, \mu_m) \to {}_m\mathrm{Br}(R) \to 1$$

Corollary (B.-K.-S.)

If $\underline{F} \cong \mathrm{Res}_{R/\mathcal{O}_S}(\mu_m)$, *then*

$$H^2_{\mathrm{fl}}(\mathcal{O}_S, \underline{F}) \cong \mathrm{Pic}(R)/m \times {}_m\mathrm{Br}(R).$$

Corollary (B.-K.-S.)

If $\underline{G}$ *is not of type A and* $\underline{F} \cong \mathrm{Res}_{R/\mathcal{O}_S}(\mu_m)$, *then*

$$H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{G}) \cong \mathrm{Pic}(R)/m \times {}_m\mathrm{Br}(R).$$

Let R be a unital commutative ring. A central exact sequence of flat R-group schemes

$$1 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 1 \qquad (3)$$

induces by flat cohomology a long exact sequence of pointed-sets

$$1 \to A(R) \to B(R) \to C(R) \to H^1_{\mathrm{fl}}(R, A) \xrightarrow{i_*} H^1_{\mathrm{fl}}(R, B) \to H^1_{\mathrm{fl}}(R, C) \qquad (4)$$

(Giraud, 1971) in which $C(R)$ acts on $H^1_{\mathrm{fl}}(R, A)$ in the following way : Given $c \in C(R)$, a preimage X of c under $B(R) \to C(R)$ is an A–bitorsor. Let $[P] \in H^1_{\mathrm{fl}}(R, A)$, then $c * [P] := [P \overset{A}{\wedge} X]$, where

$$P \overset{A}{\wedge} X := (P \times X)/(pa, a^{-1}x) \quad \forall a \in A(R).$$

Let $\Theta := \mathrm{Out}(\underline{G})$. $\underline{G}$ is reductive $\rightarrow \mathrm{Aut}(\underline{G})$ is representable as an $\mathcal{O}_S$–group scheme (SGA 3, 1962-64) and we have

$$1 \rightarrow \underline{G}^{\mathrm{ad}} \rightarrow \mathrm{Aut}(\underline{G}) \rightarrow \Theta \rightarrow 1. \tag{5}$$

We obtain

$$1 \rightarrow \underline{G}^{\mathrm{ad}}(\mathcal{O}_S) \rightarrow \mathrm{Aut}(\underline{G})(\mathcal{O}_S) \rightarrow \Theta(\mathcal{O}_S) \rightarrow H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{G}^{\mathrm{ad}}) \xrightarrow{i_*} H^1_{\mathrm{fl}}(\mathcal{O}_S, \mathrm{Aut}(\underline{G})) \rightarrow H^1_{\mathrm{fl}}(\mathcal{O}_S, \Theta) \tag{6}$$

$\Theta(\mathcal{O}_S) \curvearrowright H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{G}^{\mathrm{ad}})$:
For $P \in \Theta(\mathcal{O}_S)$, a preimage of P in $\mathrm{Aut}(\underline{G})(\mathcal{O}_S)$, regarded as a $\underline{G}^{\mathrm{ad}}$–torsor, gives rise to the inner form ${}^P\underline{G}$ of $\underline{G}$.

G is a connected reductive K−group, $\rightsquigarrow T \subset G$ a maximal torus.
Let $X(T) = \mathrm{Hom}_G(T, G_m)$ be the lattice of characters,
$\Phi = \Phi(G,T) \subset X(T) \smallsetminus \{0\}$ a set of roots of (G,T),
$R = (X, \Phi, X^\vee, \Phi^\vee)$ a root datum and
$X_{\mathbb{Q}} := X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$, a $\mathbb{Q}$−vector space with $\dim(X_{\mathbb{Q}}) = \mathrm{rank}(T)$.

Proposition (B. Conrad, 2014)

Assume Φ spans $X_{\mathbb{Q}}$ and that $(X_{\mathbb{Q}}, \Phi)$ is reduced. The inclusion $\Theta \subseteq Aut(Dyn(G))$ is an equality, if the root datum is adjoint or simply-connected, or if $(X_{\mathbb{Q}}, \Phi)$ is irreducible and $(\mathbb{Z}\Phi^\vee)^*/\mathbb{Z}\Phi$ is cyclic.

Remark

The only case of irreducible Φ in which the non-cyclicity occurs is of type $D_{2n} (n \geq 2)$, in which $(\mathbb{Z}\Phi^\vee)^*/\mathbb{Z}\Phi \cong (\mathbb{Z}/2\mathbb{Z})^2$ (Conrad, Ex. 1.5.2)

The list of all types of absolutely almost-simple K-groups (e.g. Platonov-Rapinchuk, 1994)

Type of G	$F(G^{ad})$	$\mathrm{Aut}(\mathrm{Dyn}(\underline{G}))$
${}^1\mathrm{A}_{n-1>0}$	μ_n	$\underline{\mathbb{Z}/2}$
${}^2\mathrm{A}_{n-1>0}$	$R^{(1)}_{L/K}(\mu_n)$	$\underline{\mathbb{Z}/2}$
$\mathrm{B}_n, \mathrm{C}_n, \mathrm{E}_7$	μ_2	0
${}^1\mathrm{D}_n$	$\mu_4,\ n = 2k+1$ $\mu_2 \times \mu_2,\ n = 2k$	$\underline{\mathbb{Z}/2}$
${}^2\mathrm{D}_n$	$R^{(1)}_{L/K}(\mu_4),\ n = 2k+1$ $R_{L/K}(\mu_2),\ n = 2k$	$\underline{\mathbb{Z}/2}$
${}^{3,6}\mathrm{D}_4$	$R^{(1)}_{L/K}(\mu_2)$	$\underline{S_3}$
${}^1\mathrm{E}_6$	μ_3	$\underline{\mathbb{Z}/2}$
${}^2\mathrm{E}_6$	$R^{(1)}_{L/K}(\mu_3)$	$\underline{\mathbb{Z}/2}$
$\mathrm{E}_8, \mathrm{F}_4, \mathrm{G}_2$	1	0

We have the decomposition (P. Gille, 2015)

$$H^1_{\mathrm{fl}}(\mathcal{O}_S, \mathrm{Aut}(\underline{G})) = \coprod_{[P]\in H^1_{\mathrm{fl}}(\mathcal{O}_S,\Theta)} H^1_{\mathrm{fl}}(\mathcal{O}_S, {}^P(\underline{G}^{\mathrm{ad}}))/\Theta(\mathcal{O}_S) \qquad (7)$$

where ${}^P(\underline{G}^{\mathrm{ad}})$ stands for the twisted form of $\underline{G}^{\mathrm{ad}}$ by P and the quotients are taken modulo the action of $\Theta(\mathcal{O}_S)$ on the ${}^P(\underline{G}^{\mathrm{ad}})$−torsors.

We identify representatives in $H^1_{\text{fl}}(\mathcal{O}_S, \text{Aut}(G))$ with the twisted forms of G up to $\mathcal{O}_S$-isomorphisms (Giraud, Calmès-Fasel) :

$$\text{Twist}(\underline{G}) := H^1_{\text{fl}}(\mathcal{O}_S, \text{Aut}(\underline{G})) = \{\text{twisted forms of } \underline{G}\}/\mathcal{O}_S - \text{iso}.$$

How ? The functors

$$\mathfrak{Tors}(\text{Aut}(\underline{G})) \to \mathfrak{Forms}(\underline{G}) \text{ and } \mathfrak{Forms}(\underline{G}) \to \mathfrak{Tors}(\text{Aut}(\underline{G}))$$

$$P \mapsto P \overset{\text{Aut}(\underline{G})}{\wedge} \underline{G}, \qquad \underline{H} \mapsto \text{Iso}(\underline{G}, \underline{H})$$

are an equivalence of fibered categories.

Let R be a scheme and $\boldsymbol{X_0}$ be an ***R*-form**, namely, an object of a fibered category of schemes defined over R. Let $\mathbf{Aut}_{\boldsymbol{X_0}}$ be its ***R*-group of automorphisms**. Let $\mathfrak{Forms}(\boldsymbol{X_0})$ be the **category of *R*-forms** that are locally isomorphic for some topology to X_0 and let $\mathfrak{Tors}(\mathbf{Aut}_{\boldsymbol{X_0}})$ be the **category of $\mathbf{Aut}_{\boldsymbol{X_0}}$-torsors** in that topology.

Proposition (Giraud - 1971, Calmès-Fasel - 2015)

The functors

$$\mathfrak{Tors}(Aut_{X_0}) \to \mathfrak{Forms}(X_0) \quad \text{and} \quad \mathfrak{Forms}(X_0) \to \mathfrak{Tors}(Aut_{X_0})$$

$$P \mapsto P \overset{Aut_{X_0}}{\wedge} X_0, \qquad\qquad X \mapsto Iso(X_0, X)$$

are adjoint, taking as unit and counit the maps

$$Iso(X_0, X) \overset{Aut_{X_0}}{\wedge} X_0 \to X \quad \text{and} \quad P \to Iso(X_0, (P \overset{Aut_{X_0}}{\wedge} X_0))$$

$$(\Psi, x) \mapsto \Psi(x), \quad p \mapsto (x \mapsto (p, x)),$$

and are an equivalence of fibered categories.

Proof :
For a form X of X_0, the sheaf $\mathrm{Iso}(X_0, X)$ is an Aut_{X_0} – torsor.

The counit $\mathrm{Iso}_{X_0,X} \overset{Aut_{X_0}}{\wedge} X_0 \to X$ is a local isomorphism, hence an isomorphism. One reasons equivalently with the unit $P \to \mathrm{Iso}_{X_0,(P \wedge^{Aut_{X_0}} X_0)}$. □

If $\underline{H}$ is an inner form of $\underline{G}$, then $[\underline{H}]$ belongs to $\mathrm{Im}(i_*)$ in (4), if it's an outer form to $\mathrm{coker}(i_*)$:

$$1 \to \underline{G}^{\mathrm{ad}}(\mathcal{O}_S) \to \mathrm{Aut}(\underline{G})(\mathcal{O}_S) \to \Theta(\mathcal{O}_S) \to H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{G}^{\mathrm{ad}}) \xrightarrow{i_*} H^1_{\mathrm{fl}}(\mathcal{O}_S, \mathrm{Aut}(\underline{G})) \to H^1_{\mathrm{fl}}(\mathcal{O}_S, \Theta) \quad (4)$$

We reformulate the decomposition (7) :

Proposition (B.-K.-S.)

$$\mathrm{Twist}(\underline{G}) \simeq \coprod_{[P] \in H^1_{\mathrm{fl}}(\mathcal{O}_S, \Theta)} H^1_{\mathrm{fl}}(\mathcal{O}_S, {}^P\underline{G}^{\mathrm{ad}})/\Theta(\mathcal{O}_S). \tag{8}$$

If $\underline{G}$ is almost-simple not of type A, it bijects with a disjoint union of abelian groups

$$\mathrm{Twist}(\underline{G}) \simeq \coprod_{[P] \in H^1_{\mathrm{fl}}(\mathcal{O}_S, \Theta)} H^2_{\mathrm{fl}}(\mathcal{O}_S, F({}^P\underline{G}^{\mathrm{ad}})) \tag{9}$$

iff $\Theta(\mathcal{O}_S)$ acts trivially on each component.

Corollary (B.-K.-S.)

If $\underline{G}$ is of the type $B_{n>1}$, $C_{n>1}$, E_7, E_8, F_4, G_2 for which $F(\underline{G}^{\mathrm{ad}}) \cong \underline{\mu}_m$, this yields :

$$\mathrm{Twist}(\underline{G}) \cong \mathrm{Pic}(\mathcal{O}_S)/m \times {}_m\mathrm{Br}(\mathcal{O}_S).$$

Proof :
$\Theta(\mathcal{O}_S) = 1 \to$ there is a single component with trivial action, $F(\underline{G})$ is split, hence $H^1_{\mathrm{fl}}(\mathcal{O}_S, {}^P\underline{G}^{\mathrm{ad}}) \cong \mathrm{Pic}(\mathcal{O}_S)/m \times {}_m\mathrm{Br}(\mathcal{O}_S)$.

□

Suppose $\Theta(\mathcal{O}_S) > 1$.

Example : Let A be a division $\mathcal{O}_S$-algebra of degree $n > 2$.
$\underline{G} = \underline{\mathrm{SL}}_1(A)$ and $\underline{G}^{\mathrm{op}} = \underline{\mathrm{SL}}_1(A^{\mathrm{op}})$ are inner forms of SL_n of type $\mathrm{A}_{n-1>1}$, thus $\Theta(\mathcal{O}_S) \cong \mathbb{Z}/2\mathbb{Z}$. Let $1 \neq \tau \in \Theta(\mathcal{O}_S)$. If :

$$A \cong A^{\mathrm{op}} \Leftrightarrow \tau \notin \mathrm{Aut}(\underline{G})(\mathcal{O}_S) \Leftrightarrow \mathrm{Aut}(\underline{G})(\mathcal{O}_S) \twoheadrightarrow \Theta(\mathcal{O}_S)$$

then we obtain

$$1 \mapsto \mathbb{Z}/2\mathbb{Z} \mapsto H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{G}^{\mathrm{ad}}) \xrightarrow{i_*} H^1_{\mathrm{fl}}(\mathcal{O}_S, \mathrm{Aut}(\underline{G}))$$
$$= \mathrm{H}^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{G}^{\mathrm{ad}})/\Theta(\mathcal{O}_S) \sqcup H^1_{\mathrm{fl}}(\mathcal{O}_S, {}^{\tau}\underline{G}^{\mathrm{ad}})/\Theta(\mathcal{O}_S)$$

where $\ker(i_*) = \{[\mathrm{PGL}_1(A)] = [\underline{G}^{\mathrm{ad}}], [\mathrm{PGL}_1(A^{\mathrm{op}})] = [(\underline{G}^{\mathrm{ad}})^{\mathrm{op}}]\}$.

$[\mathrm{PGL}_1(A)]$ and $[\mathrm{PGL}_1(A^{\mathrm{op}})]$ are identified in $\mathrm{Twist}(\underline{G}) \cong H^1_{\mathrm{fl}}(\mathcal{O}_S, \mathrm{Aut}(\underline{G}))$ by i_* via the inverse isomorphism $j : \underline{\mathrm{SL}}_1(A) \to \underline{\mathrm{SL}}_1(A^{\mathrm{op}}), x \mapsto x^{-1}$ defined over $\mathcal{O}_S$.

Example : Let $C = \{ Y^2 Z = X^3 + XZ^2 + Z^3 \}$ defined over $\mathbb{F}_3$.

$$C^{\mathrm{aff}} := C \setminus \{ \infty = (0:1:0) \} = \{ y^2 = x^3 + x + 1 \}$$

with $\mathcal{O}_K = \mathbb{F}_3[x,y]/(y^2 - x^3 - x - 1)$.
By Gauss' Theorem (Disquis. Arithmet. - 1801) and (Bitan, 2019)

$$cl_1'(x^3 + x + 1) \cong \mathrm{Pic}(\mathcal{O}_K) \cong \mathbb{Z}/4\mathbb{Z}$$

and with Shapiro's Lemma

$$\mathrm{Pic}(\mathcal{O}_K) \cong H^1_{\mathrm{fl}}(\mathcal{O}_K, \underline{G}_m) \cong H^1_{\mathrm{fl}}(\mathcal{O}_K, \mathrm{Aut}(\underline{G}_a))$$

we obtain

$$\mathrm{Twist}(\underline{G}_a) \cong H^1_{\mathrm{fl}}(\mathcal{O}_K, \mathrm{Aut}(\underline{G}_a)) \cong \mathbb{Z}/4\mathbb{Z}$$

Cartier duality

$$G := \mathrm{Twist}(\underline{G}_a) \to \mathrm{Twist}(\underline{G}_m) =: \mathrm{Hom}(G, K) = G^\vee,\ k \mapsto \exp\frac{2\pi ik}{n}$$

gives

$$\mathrm{Twist}(\underline{G}_m) \cong \mu_4$$

Thank you for your attention !

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