Some Current Trends in Arithmetic Dynamics[1]

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Géométrie algébrique, Théorie des nombres et Applications
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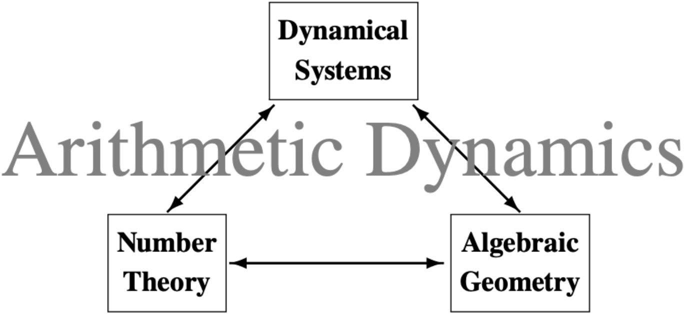
[1] R. Benedetto, P. Ingram, R. Jones, M. Manes, J. H. Silverman, and T. J. Tucker, Current trends and open problems in arithmetic dynamics, Bull. Amer. Math. Soc. (N.S.) 56 (2019), no. 4, 611–685.

Real Talk

Everything sucks. Not just for me, for a lot of us. And we don't talk about that.

PANDEMIC	university policies
moving 6K miles	no work space
quarantine	ANXIETY
FEAR	isolation
sexism	HARASSMENT
illness	elder care
attempted coup	WHY MATH
no community	no motivation

Why am I here?


Dynamical Systems
Arithmetic Dynamics
Number Theory
Algebraic Geometry

Dynamical system

Definition

A (discrete) dynamical system consists of a set S and a function $f : S \to S$. This self-mapping allows iteration.

$$f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}} = n^{\text{th}} \text{ iterate of } f$$

Definition

The (forward) **orbit** of a point $\alpha \in S$ is

$$\mathcal{O}_f(\alpha) = \{f^n(\alpha) \colon n \geq 0\}.$$

Orbits

Overarching goal of dynamics: Classify points according to their orbit structures.

$f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^2 - 1$

Start with $\alpha = 1$.

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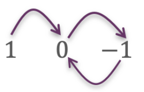

So $\alpha = 1$ is **preperiodic** under f.

$$1 \in \mathrm{PrePer}(f).$$

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So $\alpha = \varphi$ is **periodic** under f (with period 1). We also saw that 0 and -1 are periodic with period 2.

$$\{0, -1, \varphi, \bar{\varphi}\} \subset \mathrm{Per}(f) \subset \mathrm{PrePer}(f).$$

Orbits

$f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^2 - 1$

Start with α very close to $\bar{\varphi}$.

-0.618		-0.364236
-0.618076		-0.867332
-0.617982		-0.247735
-0.618098		-0.938628
-0.617955	...	-0.118978
-0.618132		-0.985844
-0.617913		-0.0281113
-0.618184		-0.99921
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This α is a **wandering point** under f with **bounded** orbit.

Orbits

$f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^2 - 1$

Start with α very close to $\bar{\varphi}$.

$-0.618034 + 1. \times 10^{-6}\, i$		$-0.765862 + 0.708062\, i$
$-0.618034 - 1.23607 \times 10^{-6}\, i$		$-0.914807 - 1.08456\, i$
$-0.618034 + 1.52786 \times 10^{-6}\, i$		$-1.33939 + 1.98432\, i$
$-0.618034 - 1.88854 \times 10^{-6}\, i$		$-3.14356 - 5.31556\, i$
$-0.618034 + 2.33437 \times 10^{-6}\, i$	...	$-19.3732 + 33.4196\, i$
$-0.618034 - 2.88544 \times 10^{-6}\, i$		$-742.545 - 1294.89\, i$
$-0.618034 + 3.5666 \times 10^{-6}\, i$		$-1.12536 \times 10^{6} + 1.92303 \times 10^{6}\, i$
$-0.618034 - 4.40856 \times 10^{-6}\, i$		$-2.43158 \times 10^{12} - 4.32821 \times 10^{12}\, i$
$-0.618034 + 5.44928 \times 10^{-6}\, i$		$-1.28208 \times 10^{25} + 2.10488 \times 10^{25}\, i$
$-0.618034 - 6.73568 \times 10^{-6}\, i$		$-2.78679 \times 10^{50} - 5.39725 \times 10^{50}\, i$

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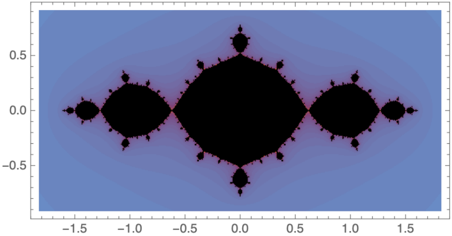
This α is a wandering point under f but the orbit is not bounded.

Classical Dynamics

Classical $\mathbb{R}$ and $\mathbb{C}$ dynamics study topological and analytic properties of orbits of points under iteration of a function on a $\mathbb{R}$ or $\mathbb{C}$ manifold.



Filled Julia set of $f(z) = z^2 - 1$.

Arithmetic Dynamics

Arithmetic dynamics studies properties of orbits of self maps with a more number-theoretic flavor.

- Maps = morphisms of algebraic varieties.
- Number-theoretic questions:
  - rational points in orbits?
  - integer points in orbits?
  - primes in orbits?
  - primitive prime divisors in orbits?

Conjugacy

Natural dynamical equivalence relation:

$$\begin{array}{ccc} S & \xrightarrow{f} & S \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ S & \xrightarrow[g]{} & S \end{array} \qquad \phi \in \mathrm{Aut}(S)$$

$$f \sim g \text{ if } g = f^{\phi} := \phi \circ f \circ \phi^{-1}.$$

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$$f \sim g \text{ if } g = f^{\phi} := \phi \circ f \circ \phi^{-1}.$$

Respects iteration:

$$(f^n)^{\phi} = \left(f^{\phi}\right)^n.$$

Lattès maps

E = elliptic curve defined over a number field K[2].

$$\begin{array}{ccc} E/K & \xrightarrow{[m]} & E/K \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle x} \\ \mathbb{P}^1 & \xrightarrow[f_{E,m}]{} & \mathbb{P}^1 \end{array}$$

$$f_{E,m} \in K(x) \text{ has degree } m^2.$$

[2]Can replace K by a local field, function field, finite field, but I'll focus on number fields for ease of exposition.

Dictionary

Elliptic Curves	**Arithmetic Dynamics**
$E \xrightarrow{\psi} E$	$f : \mathbb{P}^1 \to \mathbb{P}^1$
$\mathbb{Z}, \mathbb{Q}$ points on E	$\mathbb{Z}, \mathbb{Q}$ points in orbits

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Theorem (Siegel, 1928)

An elliptic curve $E : y^2 = x^3 + Ax + B$ with $A, B, \in \mathbb{Z}$ has only finitely many points $P = (x, y)$ with integer coordinates $x, y \in \mathbb{Z}$.

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Theorem (Silverman, 1994)

Let $f \in \mathbb{Q}(z)$ such that $f^2 \notin \mathbb{Q}[z]$. Let $\alpha \in \mathbb{Q}$. Then

$$\# \left(\mathcal{O}_f(\alpha) \cap \mathbb{Z}\right) < \infty.$$

$E_{\mathbf{tors}} \leftrightarrow \mathrm{PrePer}(f)$

$$\begin{array}{ccc} E/K & \xrightarrow{[m]} & E/K \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle x} \\ \mathbb{P}^1 & \xrightarrow[f_{E,m}]{} & \mathbb{P}^1 \end{array}$$

Suppose $P \in E_{\text{tors}}$ with $[n]P = \mathcal{O}$. The orbit of P is a finite set:

$$\{P, [m]P, [m^2]P, \ldots\} = \{P, [m \bmod n]P, [m^2 \bmod n]P, \ldots\}.$$

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$$\begin{aligned} m^k &\equiv m^\ell \pmod{n} \\ [m^k]P &= [m^\ell]P \quad \text{on } E \\ x(m^k P) &= x(m^\ell P) \\ f^k(x(P)) &= f^\ell(x(P)) \end{aligned}$$

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$$\begin{tikzcd} E/K \arrow[r, "{[m]}"] \arrow[d, "x"'] & E/K \arrow[d, "x"] \\ \mathbb{P}^1 \arrow[r, "f_{E,m}"'] & \mathbb{P}^1 \end{tikzcd}$$

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So $x(P) \in \text{PrePer}(f)$.

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Theorem (Mazur, 1977)

The torsion subgroup of $E(\mathbb{Q})$ contains at most 16 points.

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Conjecture (Morton-Silverman, 1994)

Let $[K : \mathbb{Q}] = n$ and let $f \in K(z)$ have degree $d \geq 2$. There is an absolute constant $C(d, n)$ such that

$$\#\mathrm{PrePer}(f, K) < C.$$

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torsion points Mazur's theorem	preperiodic points uniform boundedness conjecture
complex multiplication	post-critically finite maps

Postcritically finite maps

Definition

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ and let $P \in \mathbb{P}^1$. Choose a local parameter z_P at P. Then P is a **critical point** of f if

$$(df/dz_P)(P) = 0.$$

A morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ is **postcritically finite** (PCF) if every critical point of f is preperiodic. (That is, the "post-critical set" is finite.)

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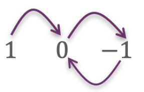
The critical points of an endomorphism f of $\mathbb{P}^1$ are the points at which f fails to be locally bijective.

Context

$f : \mathbb{P}^1 \to \mathbb{P}^1, \quad z \mapsto z^2 - 1$

Critical points: $\alpha_1 = 0$ and $\alpha_2 = \infty$. 

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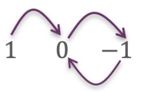
Critical points: $\alpha_1 = 0$ and $\alpha_2 = \infty$. 

Not obvious: Every Lattès map is PCF.

Hubbard: "The main question to ask about a rational map is: *what are the orbits under iteration of the critical points?*"

PCF vs CM

Some properties of CM elliptic curves:

- algebraicity (isomorphic to an EC over $\bar{\mathbb{Q}}$).
- sparsity (CM points are a countable Zariski-dense set of points in moduli space, finitely many CM curves over a given number field)
- potential good reduction at every prime (integral j invariant)
- exception to Serre's open image theorem for Galois representations

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Theorem (Thurston, 1970s)

Let $f \in \mathbb{C}(z)$ have degree $d \geq 2$. If f is PCF and not conjugate to a Lattès map, then is conjugate to one defined over $\bar{\mathbb{Q}}$.

Digression: Moduli spaces

A morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d is given by a pair of homogeneous polynomials $[f_0, f_1]$ of degree d with no common factor. Write:

$$f_0 = a_d x^d + \cdots + a_0 y^d \quad \text{and} \quad f_1 = b_d x^d + \cdots + b_0 y^d.$$

Identify the morphism f with a point in projective space:

$$f \leftrightarrow [a_d : \cdots : a_0 : b_d : \cdots : b_0] \in \underbrace{\mathbb{P}^{2d+1} \setminus \{\mathrm{Res} = 0\}}_{\mathrm{Hom}_d}.$$

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Definition

The moduli space of degree d morphisms of $\mathbb{P}^1$ is

$$M_d := \mathrm{Hom}_d \,/\, \mathrm{PGL}_2 .$$

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Theorem (Benedetto-Ingram-Jones-Levy, 2013)

Excluding the Lattès maps, PCF maps form a set of bounded height in the moduli space of rational maps of degree d.

Digression: Arboreal Galois representations

$T_n(f) =$ tree with root 0, whose collection of vertices are

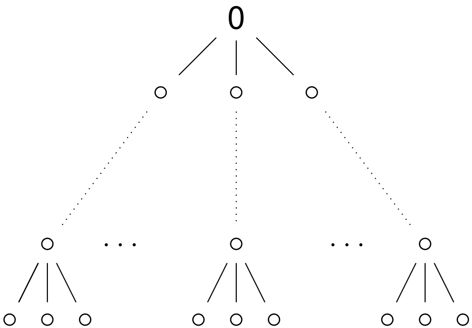
$$V_n(f) = \bigsqcup_{k=0}^{n} f^{-k}(0).$$

If $f(v) = w$, draw edge (v, w).

$T_n(f) =$ regular d-ary rooted tree for every $n \in \mathbb{N}$ provided that the forward orbits of the critical points of f avoid 0.

$T_\infty = \varprojlim_n T_n$.

Digression: Arboreal Galois representations



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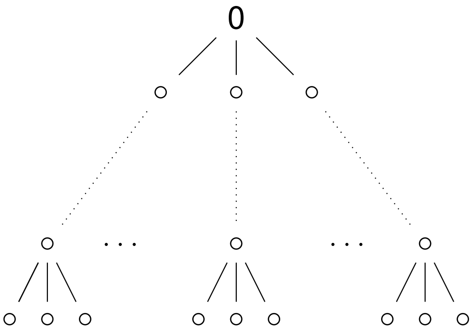

$\mathrm{Gal}_{\mathbb{Q}}$ acts on T_n by tree automorphisms.
Arboreal Galois represenation of $f(x)$:

$$\rho_f : \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{Aut}(T_\infty)$$

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Theorem (Jones, 2013)

The Galois representation associated to a PCF function of degree $d \geq 2$ has infinite index in the automorphism group of the infinite rooted d-ary tree.

Digression: (Potential) good reduction

Recall: A morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d is given by a pair of homogeneous polynomials $[f_0, f_1]$ of degree d with no common factor.

$$f_0 = a_d x^d + \cdots + a_0 y^d \quad \text{and} \quad f_1 = b_d x^d + \cdots + b_0 y^d.$$

Write $\tilde{f}_i$ for the polynomial we get by reducing the coefficients of f_i modulo a prime p.

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Definition

We say that f has **good reduction** mod p if the morphism $\tilde{f} = [\tilde{f}_0, \tilde{f}_1]$ is a morphism of degree d.

And f has **potential good reduction** if some conjugate map $g \sim f$ has good reduction at p (possibly moving to an extension field).

Digression: (Potential) good reduction

The morphism f has bad reduction at p if:

- The degree of f goes down when you reduce mod p.
- f_0 and f_1 pick up a common factor mod p.

Primes of bad reduction are precisely the primes dividing $\mathrm{Res}(f_0, f_1)$.

Potential good reduction

$f(z) = 3z^2 - 1$ has bad reduction at 3.
Conjugate by $z \mapsto z/3$:

$$3f(z/3) = 3(3z^2/9 - 1) = z^2 - 3.$$

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- sparsity (CM points are countable Zariski-dense set of points in moduli space, finitely many CM curves over a given number field) ✔
- potential good reduction at every prime (integral j invariant) ✘
- exception to Serre's open image theorem ✔

Theorem (Anderson-M-Tobin)

Let $d, p \in \mathbb{Z}$, with $d \geq 2$ and p prime. Then there exists a PCF polynomial $f \in \bar{\mathbb{Q}}[z]$ of degree d with persistent bad reduction at p if and only if $d = p^n \ell$ for some integers $n \geq 0$ and $\ell > p$ with $p \nmid \ell$.

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- potential good reduction at every prime (integral j invariant) ✘✘✘✘
- exception to Serre's open image theorem ✔

Theorem (Ingram-Krieger-Looper-M)

Let $d \geq 2$. There does not exist a finite set $S \subseteq M_{\mathbb{Q}}$ such that every PCF rational function of degree d has potential good reduction outside S.

Everything I know about unlikely intersections

Everything I know about unlikely intersections

Vaguely: “Special points” should not be dense in a variety unless the variety is “special.”

Conjecture of Lang

Proved independently by Tate, Serre, Ihara, maybe others...

A plane curve can contain only finitely many points whose coordinates are both roots of unity, unless it’s a “special curve,” that must contain infinitely many for some obvious reason.

Masser-Zannier theorem

Theorem

Let P_λ and Q_λ be linearly independent points in $E_\lambda : y^2 = x(x-1)(x-\lambda)$ (the Legendre family of elliptic curves). Then

$$\{\lambda \in \mathbb{C} \colon P_\lambda \text{ and } Q_\lambda \text{ are both torsion}\}$$

is finite.

Zannier's question

Motivated by his joint work with Masser, and by the analogy between torsion points on elliptic curves and preperiodic points for dynamical systems, Zannier posed the following question in 2008:

Question: What can be said about the set of $c \in \mathbb{C}$ such that both 0 and 1 are preperiodic for $f(z) = z^2 + c$?

- $c = 0$: 0 and 1 are both fixed
- $c = -1$: 1 maps to the two-cycle $\{0, -1\}$.
- $c = -2$: $f^2(0) = 2$ which is fixed, and $f(1) = -1$ which is fixed.

Baker and DeMarco (2009)

Theorem

Fix $a, b \in \mathbb{C}$ with $a \neq \pm b$. Then

$$\{c \in \mathbb{C} \colon \textit{a and b are both preperiodic for } z^2 + c\}$$

is finite.

The promise of arithmetic dynamics

$$\begin{array}{ccc} E/K & \xrightarrow{[m]} & E/K \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle x} \\ \mathbb{P}^1 & \xrightarrow[f_{E,m}]{} & \mathbb{P}^1 \end{array}$$

DeMarco, Krieger, Ye (2019)

Conjecture of Bogomolov, Fu, and Tschinkel

There exists a uniform constant B such that

$$\pi_1(E_1^{\text{tors}}) \cap \pi_2(E_2^{\text{tors}}) \leq B$$

for any pair of elliptic curves E_1, E_2 over $\mathbb{C}$ and any pair of standard projections π_i for which $\pi_1(E_1[2]) \neq \pi_2(E_2[2])$.

Proved for $E_{t_i} : y^2 = x(x-1)(x-t_i)$ and $\pi_i(x,y) = x$ by considering the associated Lattès maps and (heavy) machinery from arithmetic dynamics.

Thank you!