The Iwasawa Main Conjecture for modular motives

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Contents

Motivation: from $\mathbb{Z}(1)$ to T_pE

Euler (1735)

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So much work has been done on the series $\zeta(s)$ that it seems hardly likely that anything new about them may still turn up.

Euler (1737)

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Proof (Euler, 1739).

$$\left(1-2^{1-s}\right)\zeta(s) = \sum_{n=0}^{\infty}(-1)^n(n+1)^{-s} = \left(\sum_{n=0}^{\infty}(-1)^n(n+1)^{-s}x^n\right)_{x=1}$$

so (putting $s=0$ and $x=1$)

$$-\zeta(0) = \left(\sum_{n=0}^{\infty}(-1)^n x^n\right)_{x=1} = \left(\frac{1}{1+x}\right)_{x=1} = \frac{1}{2}.$$

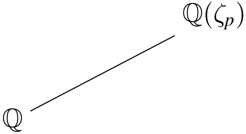
□

Dirichlet/Dedekind

1. $$\operatorname{ord}_{s=0} \zeta_{\mathbb{Q}}(s) = 0,\ \operatorname{ord}_{s=0} \zeta_K(s) = \operatorname{rank}_{\mathbb{Z}} \mathcal{O}_K^{\times}$$

2. $$\zeta_{\mathbb{Q}}(0) = -\frac{1}{2},\ \zeta_K^*(0)/\operatorname{Reg}_K = -\frac{|\operatorname{Cl}(\mathcal{O}_K)|}{|\mathcal{O}_{K,\mathrm{tors}}^{\times}|}$$

Kummer/Herbrand/Iwasawa (1969)



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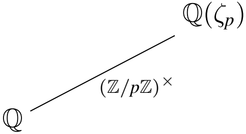

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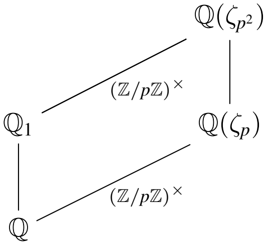

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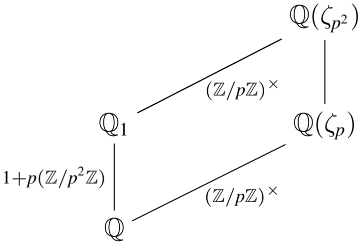

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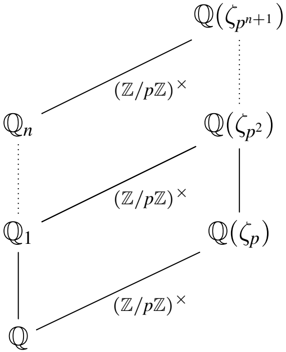

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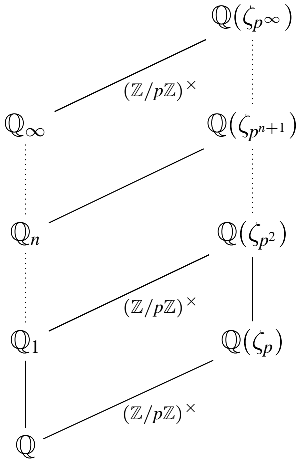

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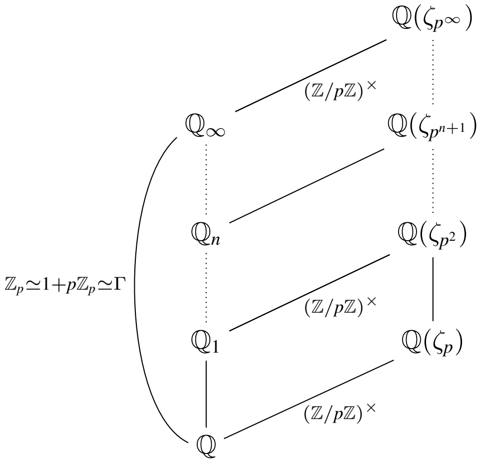

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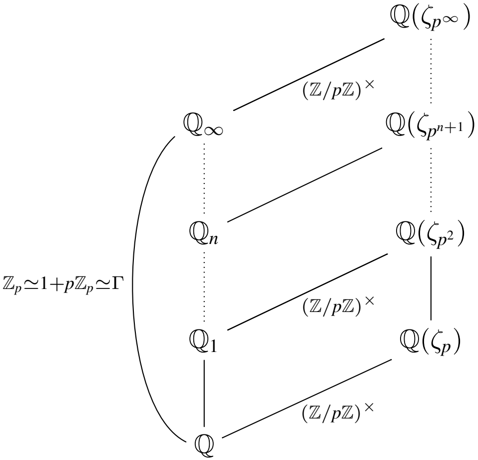
$\Lambda \stackrel{\text{def}}{=} \mathbb{Z}_p[[\Gamma]].$



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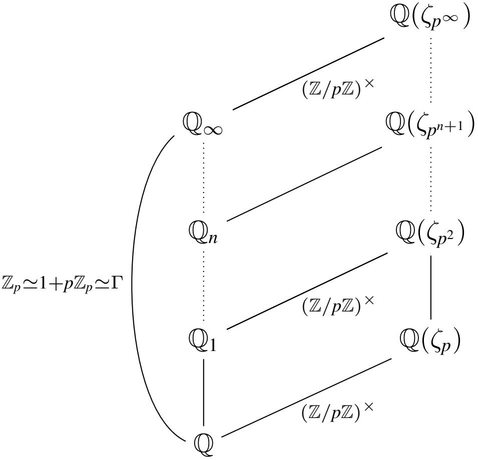
$\Lambda \stackrel{\text{def}}{=} \mathbb{Z}_p[[\Gamma]] \simeq \mathbb{Z}_p[[X]]$.



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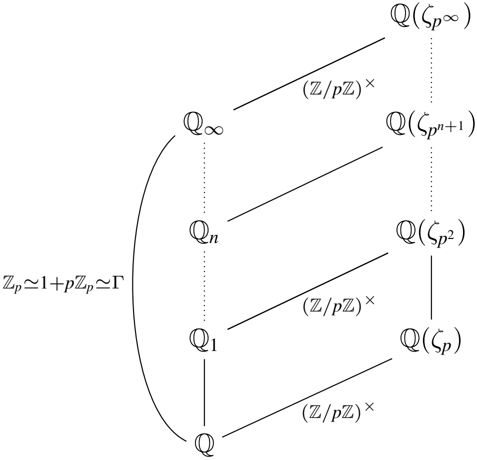

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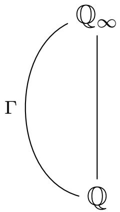

$\mathbb{Z}_p(1)_{\mathrm{Iw}} \stackrel{\text{def}}{=} \mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} \Lambda$.

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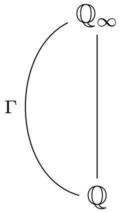

$\mathbb{Z}_p(1)_{\mathrm{Iw}} \stackrel{\text{def}}{=} \mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} \Lambda$.

$$H^i_{\mathrm{Iw}}(\mathbb{Z}_p(1)) \stackrel{\text{def}}{=} H^i_{\mathrm{et}}(\operatorname{Spec} \mathbb{Z}[1/p], \mathbb{Z}_p(1)_{\mathrm{Iw}}) \left(= H^i\left(G_{\mathbb{Q},\{p\}}, \mathbb{Z}_p(1)_{\mathrm{Iw}}\right)\right)$$

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The Λ-module $H^1_{\mathrm{Iw}}(\mathbb{Z}_p(1))$ is free of rang 1, the Λ-module $H^2_{\mathrm{Iw}}(\mathbb{Z}_p(1))$ is torsion, $H^i_{\mathrm{Iw}}(\mathbb{Z}_p(1)) = 0$ if $i \notin \{1, 2\}$.

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If M is torsion and finite-type Λ-module, it has a *characteristic* ideal

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For instance, the characteristic ideal of $\Lambda/X \oplus \Lambda/X \oplus \Lambda/p^2$ is $p^2X^2\Lambda$.

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$$\mathrm{char}_\Lambda H^2_{\mathrm{Iw}}(\mathbb{Z}_p(1)) = \mathrm{char}_\Lambda H^1_{\mathrm{Iw}}(\mathbb{Z}_p(1))/\Lambda \cdot \mathbf{z}.$$

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Define

$$\Delta(\mathbb{Z}_p(1)) \overset{\mathrm{def}}{=} \mathrm{Det}_\Lambda^{-1} \mathrm{R}\,\Gamma_{\mathrm{et}}(\mathbb{Z}[1/p],\mathbb{Z}_p(1)_{\mathrm{Iw}}) \otimes_\Lambda \mathrm{Det}_\Lambda^{-1} \Lambda^+.$$

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Iwasawa's conjecture is then the statement that the following diagram is commutative

$$\begin{array}{ccc}
\Delta(\mathbb{Z}_p(1)) & \xrightarrow{\ \simeq\ } & \Lambda \\
\downarrow & & \downarrow \\
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\end{array}$$

The case $\mathbb{Z}(1)$

1. $\operatorname{ord}_{s=0} \zeta_K(s) = \operatorname{rank}_{\mathbb{Z}} \mathcal{O}_K^{\times}$.
2. There are formulas for $\zeta_{\mathbb{Q}}(0)$ and $L(1-n,\chi)$.
3. $\operatorname{char}_\Lambda H^2_{\mathrm{Iw}}(\mathbb{Z}_p(1)) \overset{?}{=} \operatorname{char}_\Lambda H^1_{\mathrm{Iw}}(\mathbb{Z}_p(1))/\Lambda\cdot\mathbf{z}$.
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3 holds (Mazur-Wiles,1984). 3 and 4 are equivalent, 4 and 1 imply 2 (Coates-Lichtenbaum,Kato).

Elliptic curves

We do the same thing for T_pE, the Tate module of a rational elliptic curve $E/\mathbb{Q}$.

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$L(E,s)$ has a holomorphic continuation to $\mathbb{C}$ (conjectured by Hasse and proved by Wiles).

Elliptic curves

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1. $\operatorname{ord}_{s=1} L(E,s)$?
2. Formulas for $L(E,1)$ and $L(E,\chi,1)$?
3. $\operatorname{char}_\Lambda H^2_{\mathrm{Iw}}(T_pE) \stackrel{?}{=} \operatorname{char}_\Lambda H^1_{\mathrm{Iw}}(T_pE)/\Lambda \cdot \mathbf{z}$ (for an appropriate $\mathbf{z}$ linked to the previous formulas).
4. $$\begin{array}{ccc} \Delta(T_pE) & \xrightarrow{\simeq} & \Lambda \\ \big\downarrow & & \big\downarrow \\ \Delta(T_pE) \otimes_\Lambda Q(\Lambda) & \xrightarrow{\mathbf{z}} & Q(\Lambda) \end{array}$$

Elliptic curves

1. $\operatorname{ord}_{s=1} L(E,s) \stackrel{?}{=} \operatorname{rank}_{\mathbb{Z}} E(\mathbb{Q})$. **The Birch and Swinnerton-Dyer Conjecture**.
2. Formulas for $L(E,1)$ and $L(E,\chi,1)$. **The Birch and Swinnerton-Dyer Conjecture**.

$$L^*(E,1)/\Omega_E \operatorname{Reg}_E = \frac{|Ш(E/\mathbb{Q})| \prod_{\ell} \operatorname{Tam}_\ell(E/\mathbb{Q})}{|E(\mathbb{Q})_{\mathrm{tors}}|^2}$$

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Kato showed that 3 and 4 are equivalent and that 4 and $L(E,1)\neq 0$ implied 2.

Modular forms

Let $f \in S_k(\Gamma_0(N))$ be an eigencuspform of weight $k \geq 2$.

1. $\operatorname{ord}_{s=1} L(f,s)$?
2. Formulas for $L(f,1)$ and $L(f,\chi,1)$.
3. $\operatorname{char}_\Lambda H^2_{\mathrm{Iw}}(\rho_f) \stackrel{?}{=} \operatorname{char}_\Lambda H^1_{\mathrm{Iw}}(\rho_f)/\Lambda \cdot \mathbf{z}(f)$. For $\mathbf{z}(f)$ constructed by Kato, **the Iwasawa Main Conjecture**.
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The main result

Elliptic curves

Theorem (with X.Wan)

Let $E/\mathbb{Q}$ be an elliptic curve with conductor N. Let $p > 2$ be a prime. Assume that $E[p]$ is a irreducible Galois representation. Assume that there exists $\ell || N, \ell \nmid p$ such that $\dim_{\mathbb{F}_p} E[p]^{I_\ell} = 1$ and $\dim_{\mathbb{F}_p} E[p]^{G_{\mathbb{Q}_\ell}} = 0$.

1. $\operatorname{ord}_{s=1} L(E,s) = 0$ *if and only if* $\operatorname{rank}_{\mathbb{Z}} E(\mathbb{Q}) = 0$.
2. *If $L(E,1) \neq 0$, then the Birch and Swinnerton-Dyer Conjecture at p is true*

$$v_p\left(L(E,1)/\Omega_E\right) = v_p\left(|Ш(E/\mathbb{Q})[p^\infty]|\prod_q \operatorname{Tam}_q(E/\mathbb{Q})\right).$$

3. *The Iwasawa Main Conjecture holds for E, that is to say the diagram*

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is commutative.

Modular forms

Theorem (with X.Wan)

Let $f \in S_k(\Gamma_0(N))$ be an eigencuspform of weight $k \geq 2$. Let $p > 2$ be a prime. Assume

1. $\bar{\rho}_f$ *is absolutely irreducible.*
2. $\left(\bar{\rho}_f | G_{\mathbb{Q}_p}\right)^{ss}$ *is neither* $\chi \oplus \chi$ *nor* $\chi \oplus \chi\chi_{\mathrm{cyc}}$.
3. *There exists* $\ell || N, \ell \nmid p$ *such that* $\dim_{\mathbb{F}_q} \bar{\rho}_f^{I_\ell} = 1$ *and* $\dim_{\mathbb{F}_q} \bar{\rho}_f^{G_{\mathbb{Q}}} = 0$.

Then the Iwasawa Main Conjecture holds for f, that is to say the diagram

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Comparison with earlier results

Theorem (Kato, Skinner-Urban, Wan, Sprung...)

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4. *Either* $k = 2$ *and* $p \nmid N$, *or* $\rho_f | G_{\mathbb{Q}_p}$ *is reducible.*

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4. *Either* $\rho_f|G_{\mathbb{Q}_p}$ *is reducible or* $k = 2$ *and* $p \nmid N$.

Then the Iwasawa Main Conjecture holds for f.

Compared to previous results, we have eliminated the last hypothesis on $\rho_f|G_{\mathbb{Q}_p}$.

Comparison with earlier results

Theorem (Kato, Skinner-Urban, Wan, Sprung...)

Let $f \in S_k(\Gamma_0(N))$ be an eigencuspform of weight $k \geq 2$. Let $p > 2$ be a prime. Assume

1. *$\bar{\rho}_f$ is absolutely irreducible.*
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Then the Iwasawa Main Conjecture holds for f.

Compared to previous results, we have eliminated the last hypothesis on $\rho_f|G_{\mathbb{Q}_p}$. From the point of view of p-adic Hodge theory, $\rho_f|G_{\mathbb{Q}_p}$ reducible or $k = 2$ and $p \nmid N$ (Fontaine-Lafaille in weight 2) are the simplest $G_{\mathbb{Q}_p}$-representations possible.

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Another way to quantify is to fix p and $\bar{\rho} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{F}_q)$ and ask whether the modular forms f such that $\bar{\rho}_f \simeq \bar{\rho}$ typically satisfy the hypothesis we eliminate.

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The answer is that the set of modular forms which satisfy it is either empty or has large codimension.

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The proof proceeds in four steps.

1. Embed ρ_f as a point in a space called the universal deformation ring of $\bar{\rho}_f$ and show that $\mathbf{z}(f) \in \Delta(\rho_f)$ interpolate in this space.
2. Show that the Iwasawa Main Conjecture is true at some subset of points in this space.

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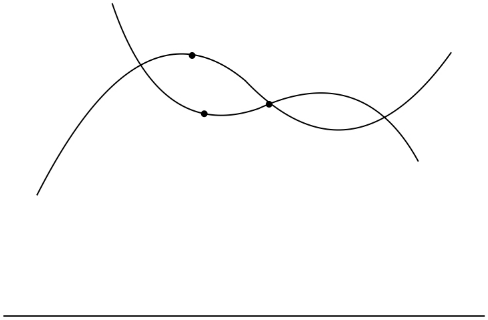

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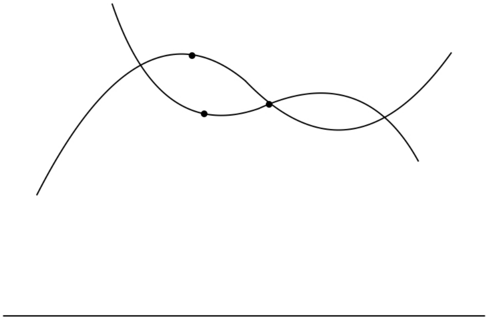
2. Show that the Iwasawa Main Conjecture is true at some subset of points in this space. We show the IMC at the set of so-called short, crystalline points. In particular, they are unramified over the base-space.



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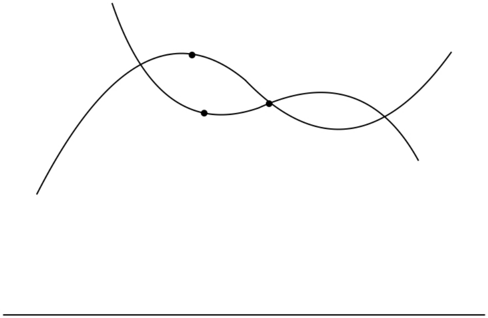
3. Deduce that it is true generically, that is to say outside a set of large codimension. We show that the fact that the IMC is true at the short, crystalline points entails that it is true outside the points of ramification (and other difficulties which we know don't occur for our original point).



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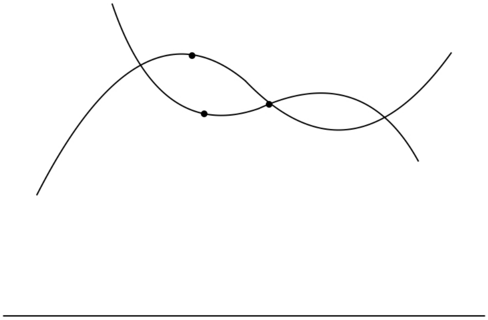
4. Deduce that it is true at the original point corresponding to ρ_f. We use a limit argument to show the IMC at our original point by moving closer and closer to it while staying in our good set.



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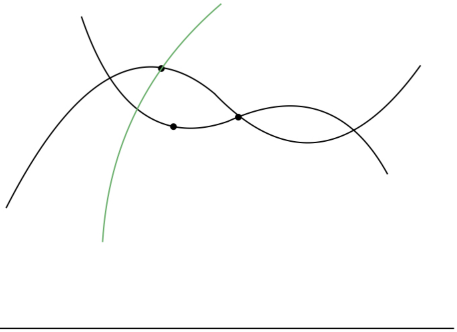
An amusing feature of the proof, is that in te end we prove something on the green family below, very far from the bold point, which is itself very far from the point we are originally interested in.



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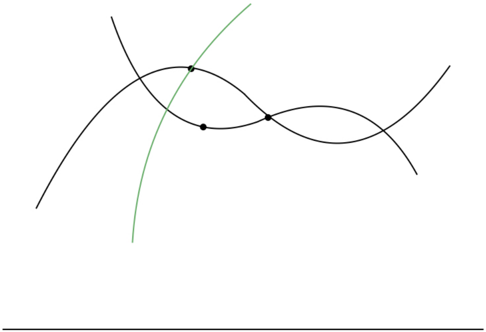
An amusing feature of the proof, is that in te end we prove something on the green family below, very far from the bold point, which is itself very far from the point we are originally interested in. But that's Euler again.



The main result

Theorem (with X.Wan)

Let $f \in S_k(\Gamma_0(N))$ be an eigencuspform of weight $k \geq 2$. Let $p > 2$ be a prime. Assume

1. *$\bar{\rho}_f$ is absolutely irreducible.*
2. *$\left(\bar{\rho}_f | G_{\mathbb{Q}_p}\right)^{ss}$ is neither $\chi \oplus \chi$ nor $\chi \oplus \chi\chi_{\mathrm{cyc}}$.*
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Then the Iwasawa Main Conjecture holds for f, that is to say the diagram

$$\begin{array}{ccc} \Delta(\rho_f) & \xrightarrow{\simeq} & \Lambda \\ \big\downarrow & & \big\downarrow \\ \Delta(\rho_f) \otimes_\Lambda Q(\Lambda) & \xrightarrow{\mathbf{z}(f)} & Q(\Lambda) \end{array}$$

is commutative. In particular, if $E/\mathbb{Q}$ satisfies the hypothesis of the theorem $L(E,1) = 0$ if and only if $E(\mathbb{Q})$ is a torsion group and in that case the Birch and Swinnerton-Dyer Conjecture holds at p.