Omega results for cubic field counts via the Katz-Sarnak philosophy

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20 August 2021
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Definition

We say that K is a cubic field if K is an extension of $\mathbb{Q}$ of degree 3.

Question

How many cubic fields of discriminant up to X are there, up to isomorphism ?

Theorem (Davenport-Heilbronn, 1971)

Counting up to isomorphism,

$$\mathcal{N}^{+}(X) := \#\{K \text{ cubic}, 0 < D_K < X\} \sim \frac{X}{12\zeta(3)};$$

$$\mathcal{N}^{-}(X) := \#\{K \text{ cubic}, 0 < -D_K < X\} \sim \frac{\sqrt{3}X}{12\zeta(3)}.$$

Theorem (Belabas, 1999)

$$\mathcal{N}^+(X) = \frac{X}{12\zeta(3)} + O(Xe^{-c(\log X \log\log X)^{\frac{1}{2}}}),$$

$$\mathcal{N}^-(X) = \frac{\sqrt{3}X}{12\zeta(3)} + O(Xe^{-c(\log X \log\log X)^{\frac{1}{2}}}).$$

Belabas also developed a fast algorithm to count cubic fields, and performed extensive numerics.

Belabas, Bhargava and Pomerance

This was improved to a power saving estimate.

Theorem (Belabas, Bhargava and Pomerance, 2010)

$$\mathcal{N}^{+}(X) = \frac{X}{12\zeta(3)} + O_{\varepsilon}(X^{\frac{7}{8}+\varepsilon}),$$

and similarly for $\mathcal{N}^{-}(X)$.

More generally, Yang used their arguments to deduce that

$$\begin{aligned} N_p^{\pm}(X,T) &:= \#\{K \in \mathcal{F}^{\pm}(X) : p \text{ has splitting type } T \text{ in } K\} \\ &= A_p^{\pm}(T)X + O_{\varepsilon}(p^2 X^{\frac{7}{8}+\varepsilon}) \end{aligned}$$

as well as application on low-lying zeros of Dedekind zeta functions.

Roberts's conjecture

Most cubic fields are non-Galois. Indeed, Cohn (1954) showed that

$$\#\{K \text{ cubic Galois}, 0 < D_K < X\} \sim CX^{\frac{1}{2}},$$

so from now on we will count

$$N^{\pm}(X) := \#\{K \text{ cubic non-Galois}, 0 < \pm D_K < X\}$$

From Belabas's numerical data, Roberts came up with the following conjecture.

Conjecture (Roberts, 2000)

$$N^{+}(X) = \frac{X}{12\zeta(3)} + \frac{4\zeta(\frac{1}{3})}{5\Gamma(\frac{2}{3})^3\zeta(\frac{5}{3})}X^{\frac{5}{6}} + o(X^{\frac{5}{6}}),$$

and similarly for $N^{-}(X)$.

Roberts's conjecture

This conjecture was proven true by Bhargava, Shankar and Tsimerman.

Theorem (Bhargava, Shankar and Tsimerman, 2013)

$$N^{+}(X) = \frac{X}{12\zeta(3)} + \frac{4\zeta(\frac{1}{3})}{5\Gamma(\frac{2}{3})^{3}\zeta(\frac{5}{3})}X^{\frac{5}{6}} + O_{\varepsilon}(X^{\frac{5}{6}-\frac{1}{48}+\varepsilon}),$$

and similarly for $N^{-}(X)$.

Taniguchi and Thorne

The error term was improved by Taniguchi and Thorne using Shintani zeta functions, which have poles at $s = 1$ and $s = \frac{5}{6}$.

Theorem (Taniguchi and Thorne, 2013)

$$N^+(X) = \frac{X}{12\zeta(3)} + \frac{4\zeta(\frac{1}{3})}{5\Gamma(\frac{2}{3})^3\zeta(\frac{5}{3})} X^{\frac{5}{6}} + O_\varepsilon(X^{\frac{7}{9}+\varepsilon}),$$

and similarly for $N^-(X)$.

More generally, they showed that

$$\begin{aligned} N_p^{\pm}(X, T) &= \#\{K \in \mathcal{F}^{\pm}(X) : p \text{ has splitting type } T \text{ in } K\} \\ &= A_p^{\pm}(T)X + B_p^{\pm}(T)X^{\frac{5}{6}} + O_\varepsilon(p^{\frac{16}{9}} X^{\frac{7}{9}+\varepsilon}). \end{aligned}$$

Finally, Bhargava, Taniguchi and Thorne (2021) obtained the best known error term for this problem, $O_\varepsilon(X^{\frac{2}{3}+\varepsilon})$. More generally, they proved :

Theorem (Bhargava, Taniguchi and Thorne, 2021)

$$\begin{aligned} N_p^\pm(X,T) &= \#\{K \in \mathcal{F}^\pm(X) : p \text{ has splitting type } T \text{ in } K\} \\ &= A_p^\pm(T)X + B_p^\pm(T)X^{\frac{5}{6}} + O_\varepsilon(p^{\frac{2}{3}}X^{\frac{2}{3}+\varepsilon}). \end{aligned}$$

The counting estimates of Belabas, Bhargava, Pomerance, Shankar, Tsimerman, Taniguchi and Thorne have applications to low-lying zeros of Dedekind zeta functions (see Yang 2009, Cho and Kim 2015, Shankar, Södergren and Templier 2019, 2021).

Best possible error term ?

Question : what is the best possible error term for $N_p^{\pm}(X,T)$?

Theorem (Cho, F., Lee, Södergren, 2021)

Assume the Riemann Hypothesis for Dedekind zeta functions. If

$$N_p^{\pm}(X,T) = A_p^{\pm}(T)X + B_p^{\pm}(T)X^{\frac{5}{6}} + O_{\varepsilon}(p^{\omega}X^{\theta}),$$

then $\omega + \theta \geq \frac{1}{2}$.

In other words, an error term of the form $O_{\varepsilon}(p^{\frac{1}{4}}X^{\frac{1}{4}-\varepsilon})$ would contradict the Generalized Riemann Hypothesis.

Ideas of the proof

- Estimates on cubic field counts imply estimates on low-lying zeros of Dedekind zeta functions (in the sense of Katz-Sarnak).
- The Katz-Sarnak prediction has a transition, which takes the shape of a lower-order term.
- If the counting estimate is too good, then this term is too large, contradicting the RH.

Cubic field counts and low-lying zeros

For a cubic field K, the one-level density is

$$\mathfrak{D}_\phi(K) := \sum_{\gamma_K} \phi\left(\frac{\log(X/(2\pi e)^2)}{2\pi}\gamma_K\right),$$

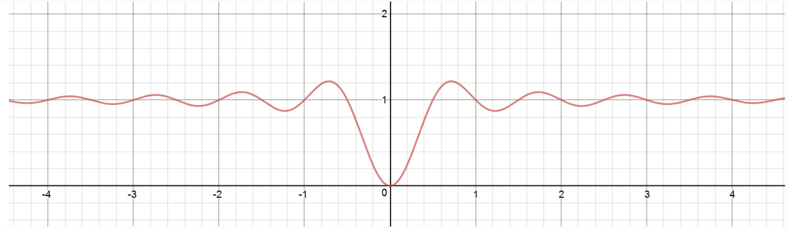
where γ_K runs over the imaginary parts of the zeros of $\zeta_K(s)$ and ϕ is a test function. The Katz-Sarnak prediction :

$$\lim_{X\to\infty} \frac{1}{N^\pm(X)} \sum_{\substack{K \text{ cubic} \\ 0<\pm D_K<X}} \mathfrak{D}_\phi(K) = \int_{\mathbb{R}} \left(1 - \frac{\sin(2\pi t)}{2\pi t}\right)\phi(t)dt.$$

"Symplectic symmetry type" (i.e. distributed like eigenvalues of large symplectic matrices)

Repulsion at 0

A graph of $1 - \frac{\sin(2\pi x)}{2\pi x}$.



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where γ_K runs over the imaginary parts of the zeros of $\zeta_K(s)$ and ϕ is an even test function. The Katz-Sarnak prediction :

$$\lim_{X\to\infty} \frac{1}{N^\pm(X)} \sum_{\substack{K \text{ cubic} \\ 0<\pm D_K<X}} \mathfrak{D}_\phi(K) = \int_{\mathbb{R}} \left(1 - \frac{\sin(2\pi t)}{2\pi t}\right)\phi(t)dt$$

$$= \widehat{\phi}(0) - \frac{1}{2}\int_{-1}^{1} \widehat{\phi}(\xi)d\xi.$$

Note 1 : if $\widehat{\phi}$ is supported in $(-1,1)$, then this is equal to $\phi(0) - \frac{\phi(0)}{2}$.

Note 2 : trivially, the one-level density is $\ll \log X$.

Idea : a "too good" counting function error term would contradict the Katz-Sarnak prediction !

Explicit formula : ($L := \log(X/(2\pi e)^2)$)

$$\mathfrak{D}_\phi(K) = \sum_{\gamma_K} \phi\Big(\frac{L\gamma_K}{2\pi}\Big) = \frac{\widehat{\phi}(0)}{L} \log|D_K| + \frac{1}{\pi}\int_{-\infty}^{\infty} \phi\Big(\frac{Lr}{2\pi}\Big) \Re\Big(\frac{\Gamma'_\pm}{\Gamma_\pm}(\tfrac{1}{2}+ir)\Big) dr$$

$$- \frac{2}{L}\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \widehat{\phi}\Big(\frac{\log n}{L}\Big) a_K(n),$$

where

$$-\frac{\zeta_K'(s)}{\zeta_K(s)} = \sum_{n\geq 1} \frac{a_K(n)}{n^s}.$$

Cubic field counts and low-lying zeros

The next step is to average this formula over cubic fields. The first term averages to $\widehat{\phi}(0)$. The second is small. For the third term, we need to understand $a_K(p^e)$, which depends on the splitting type of p :

Splitting type	(p)	$a_K(p^e)$
T_1	$\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$	2
T_2	$\mathfrak{p}_1\mathfrak{p}_2$	$1+(-1)^e$
T_3	$\mathfrak{p}_1$	η_e
T_4	$\mathfrak{p}_1^2\mathfrak{p}_2$	1
T_5	$\mathfrak{p}_1^3$	0,

$$\eta_e := \begin{cases} 2 & \text{if } e \equiv 0 \bmod 3; \\ -1 & \text{if } e \equiv \pm 1 \bmod 3. \end{cases}$$

The third term is computed as follows :

$$\sum_{\substack{K \text{ cubic} \\ 0<\pm D_K<X}} a_K(p^e) = 2N_p^{\pm}(X, T_1) + (1+(-1)^e)N_p^{\pm}(X, T_2) + \eta_e N_p^{\pm}(X, T_3) + N_p^{\pm}(X, T_4)$$

Cubic field counts and low-lying zeros

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$$\sum_{\substack{K \text{ cubic} \\ 0<\pm D_K<X}} a_K(p^e) = 2N_p^{\pm}(X,T_1) + (1+(-1)^e)N_p^{\pm}(X,T_2) + \eta_e N_p^{\pm}(X,T_3) + N_p^{\pm}(X,T_4)$$

Hence, assuming an error term of $O_\varepsilon(p^\omega X^{\theta+\varepsilon})$ in cubic field counts, this is

$$= C_1^{\pm} X(\theta_e + \tfrac{1}{p})x_p + C_2^{\pm} X^{\frac{5}{6}}(1+p^{-\frac{1}{3}})(\kappa_e(p) + p^{-1} + p^{-\frac{4}{3}})y_p + O_\varepsilon(p^\omega X^{\theta+\varepsilon}),$$

where $C_j^{\pm}$ are constants, $x_p := (1+\frac{1}{p}+\frac{1}{p^2})^{-1}$,
$y_p := \frac{1-p^{-\frac{1}{3}}}{(1-p^{-\frac{5}{3}})(1+p^{-1})}$, and $\theta_e, \kappa_e(p)$ are of similar nature.

After some calculations, the sum of all three terms is shown to equal

$$\widehat{\phi}(0) - \frac{\phi(0)}{2} - \frac{2C_2^{\pm}X^{-\frac{1}{6}}}{C_1^{\pm}}\int_0^\infty \widehat{\phi}(u)\Big(\frac{X}{(2\pi e)^2}\Big)^{\frac{u}{6}} du + o(1).$$

Recall the Katz-Sarnak prediction :

$$\widehat{\phi}(0) - \frac{1}{2}\int_{-1}^{1}\widehat{\phi}(\xi)d\xi,$$

which is equal to $\widehat{\phi}(0) - \frac{\phi(0)}{2}$ when $\widehat{\phi}$ is supported in $(-1,1)$. Thus we obtain a contradiction as soon as $\widehat{\phi}$ has support outside $(-1,1)$, since the RH implies that the one-level density is $\ll \log X$.

Other results

- We study the Ratios Conjecture recipe of Conrey, Farmer and Zirnbauer, which is supposed to give an estimate with error term $O_\varepsilon(X^{-\frac{1}{2}+\varepsilon})$.
- We find that the original recipe gives precision $O(X^{-\frac{1}{6}})$, at best.
- We refine the recipe to one which gives precision $O(X^{-\frac{1}{3}})$ (conjecturally).
- Assuming a slightly better estimate than Bhargava-Taniguchi-Thorne (any power $< \frac{2}{3}$ would work), we can show that the error $O(X^{-\frac{1}{3}})$ is best possible in this family.

Thank you !